天体力学II講義ノート

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1　正準運動方程式と正準変換

1.1　正準運動方程式

ハミルトニアン $F = F(x_1, \cdots, x_n; y_1, \cdots, y_n; t) = F(x, y; t)$

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \quad (i = 1, \cdots n) \tag{1}$$

ここで，x_i を運動量，y_i を座標と呼ぶことにする．この方程式は $2n$ 階の微分方程式である．$x = x(t), y = y(t)$ と考えて，

$$\frac{dF}{dt} = \frac{\partial F}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial F}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial F}{\partial t} \tag{2}$$

特に，$x = x(t), y = y(t)$ が (1) の解なら，

$$\frac{dF}{dt} = \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial y_i} - \frac{\partial F}{\partial y_i}\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial t} \tag{3}$$

F が t を explicit に含まなければ，

$$\frac{\partial F}{\partial t} = 0$$

より，

$$F(x(t), y(t)) = \text{const.} = C, \quad (\text{第一積分}) \tag{4}$$

となる．このとき (1) は $2n - 2$ 階の系と 1 つの求積となる．実際，

$$\begin{cases} \frac{dx_2}{dx_1} = \frac{\frac{\partial F}{\partial y_2}}{\frac{\partial F}{\partial y_1}}, \cdots, \frac{dy_1}{dx_1} = -\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial y_1}}, \cdots \\ F(x_1, \cdots, y_1, \cdots) = C \end{cases} \tag{5}$$

これは $2n-2$ 階の微分方程式．これを解くことによって，$x_2, \cdots, y_n$ が x_1 と C と $2n-2$ 個の積分定数で表せる．このとき求積，

$$t = \int \frac{dx_1}{\frac{\partial F}{\partial y_1}} + c \tag{6}$$

結局，積分定数は $2n$ 個．
特に $n = 1$ の時は，

$$\frac{dx}{dt} = \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial F}{\partial x}, \quad F = F(x, y) \tag{7}$$

$$\left.\begin{array}{l} \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \\ F(x.y) = C \end{array}\right\} \quad 0\,\text{階の系} \quad F(x,y) = C \quad \rightarrow \quad y = y(x,C) \tag{8}$$

これより直ちに，求積，

$$t = \int \frac{dx}{\frac{\partial F}{\partial y}} + c \tag{9}$$

例：

$$\left[\begin{array}{l} F = \frac{1}{2}y^2 + U(x) \\ y = \sqrt{2(C - U(x))} \\ t = \int \frac{dx}{\sqrt{2(C-U(x))}} + c \end{array}\right. \tag{10}$$

1.2　正準運動方程式と正準変換 ($\frac{\partial F}{\partial t} = 0$ の場合の続き)

$$\frac{dx_i}{dt} - \frac{\partial F}{\partial y_i} \equiv E_i, \quad \frac{dy_i}{dt} + \frac{\partial F}{\partial x_i} \equiv E_{n+i}, \quad (i = 1 - n) \tag{11}$$

とおくと，

$$(1) : F_j = 0, \quad (j = \underline{1} - 2n) \tag{12}$$

さて，これは，

$$F = C, \quad E_j = 0, \quad (j = \underline{2} - 2n) \tag{13}$$

と同じであろうか？

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial x_i}\left(\frac{dx_i}{dt} - \frac{\partial F}{\partial y_i}\right) + \frac{\partial F}{\partial y_i}\left(\frac{dy_i}{dt} + \frac{\partial F}{\partial x_i}\right) \\ &= \frac{\partial F}{\partial x_i}E_i + \frac{\partial F}{\partial y_i}E_{n+i} \end{aligned} \tag{14}$$

まず，x, y が (12) を満たすとすると，

$$\frac{dF}{dt} = 0 \quad \Rightarrow \quad F = C, \quad (13)\,\text{が出る} \tag{15}$$

次に (13) を満たすとすると，

$$0 = \frac{\partial F}{\partial x_1}E_1 \quad \Rightarrow \quad E_1 = 0 \quad \text{if} \quad \frac{\partial F}{\partial x_1} \neq 0 \tag{16}$$

もし，$\frac{\partial F}{\partial x_1} \equiv 0$ なら一般に E_1 はゼロではない．

$$\frac{dy_1}{dt} = 0 \quad \Rightarrow \quad y_1 = \text{const.} \tag{17}$$

220.110.46.64 よって階数が 2 階落ちたものと同じ．

よって (12) は常に (13) と等しい (上の意味で)．よって積分が求まれば次々に階数を減らすことができる．

1.3 正準変換の条件

今，F は t を explicit に含まない場合を考えている．

$$\exists \text{変換}\ \mathrm{T} : x, y \to x', y'$$

によって (1) が

$$\frac{dx'_i}{dt} = \frac{\partial F^*}{\partial y'_i}, \quad \frac{dy'_i}{dt} = -\frac{\partial F^*}{\partial x'_i} \tag{18}$$

$$F^* = F(x(x', y'), y(x', y')) \tag{19}$$

なら T を正準変換 (Canonical Transformation) と言う．
x', y' を $u_j (j = 1 - 2n)$ と書く．

1. x, y は (1) を満たす：

$$\begin{aligned}
\frac{du_i}{dt} &= \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dt} + \frac{\partial u_i}{\partial y_j}\frac{dy_j}{dt} \\
&= \frac{\partial u_i}{\partial x_j}\frac{\partial F}{\partial y_j} - \frac{\partial u_i}{\partial y_j}\frac{\partial F}{\partial x_j} \\
&= \frac{\partial u_i}{\partial x_j}\frac{\partial F^*}{\partial u_k}\frac{\partial u_k}{\partial y_j} - \frac{\partial u_i}{\partial y_j}\frac{\partial F^*}{\partial u_k}\frac{\partial u_k}{\partial x_j} \\
&= \frac{\partial F^*}{\partial u_k}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial y_j} - \frac{\partial u_k}{\partial y_j}\frac{\partial u_i}{\partial x_j}\right) \\
\frac{du_i}{dt} &= \frac{\partial F^*}{du_k}P_{ik}, \quad (i, k = 1 - 2n)
\end{aligned} \tag{20}$$

ここで，

$$P_{ik} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial y_j} - \frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial y_j} \quad \text{Poisson bracket} \tag{21}$$

明らかに，

$$P_{ik} = -P_{ki}, \quad \text{Antisym.} \quad \text{独立な成分は}\, 2(2n-1) \tag{22}$$

(20) が正準方程式になるためには，適当な変換の後に，

$$\frac{du_i}{dt} = \frac{\partial F^*}{du_{n+1}}, \quad \frac{du_{n+1}}{dt} = -\frac{\partial F^*}{du_i}, \quad (i = 1 - n) \tag{23}$$

つまり，

$$P_{ik} = \delta_{i,k-n} - \delta_{i,k+n} = \begin{pmatrix} \mathbf{0} & \boldsymbol{i} \\ -\boldsymbol{i} & \mathbf{0} \end{pmatrix} \tag{24}$$

このようなテンソル P は正準変換を持つという．

(a) $n=1$ の時，運動方程式は (7)．このとき

$$P_{ik} = \begin{pmatrix} 0 & \frac{\partial(u_1,u_2)}{\partial(x,y)} \\ \frac{\partial(u_2,u_1)}{\partial(x,y)} & 0 \end{pmatrix} \tag{25}$$

正準変換を持つためには，

$$\frac{\partial(u_1,u_2)}{\partial(x,y)} = 1 \tag{26}$$

問

$$\begin{cases} x' = Z(x)\cos y \\ y' = Z(x)\sin y \end{cases}$$

は $Z(x)$ を適当にとって正準変換となるか？

解

$$\begin{aligned} \frac{\partial(x',y')}{\partial(x,y)} &= \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial x} \\ &= ZZ'(\cos^2 y + \sin^2 y) = ZZ' = 1 \\ Z^2 &= 2(x+C), \quad Z = \sqrt{2(x+C)} \end{aligned} \tag{27}$$

$C=0$ とおいて，

$$x' = \sqrt{2x}\cos y, \quad y' = \sqrt{2x}\sin y \tag{28}$$

これは正準変換．これを Poincaré の変換という．

2.

$$\begin{aligned} \frac{\partial F^*}{\partial u_k} &= \frac{\partial F}{\partial x_j}\frac{\partial x_j}{\partial u_k} + \frac{\partial F}{\partial y_j}\frac{\partial y_j}{\partial u_k} \\ &= -\frac{dy_j}{dt}\frac{\partial x_j}{\partial u_k} + \frac{dx_j}{dt}\frac{\partial y_j}{\partial u_k} \\ &= -\frac{\partial y_j}{\partial u_i}\frac{du_i}{dt}\frac{\partial x_j}{\partial u_k} + \frac{\partial x_j}{\partial u_i}\frac{du_i}{dt}\frac{\partial y_j}{\partial u_k} \\ &= \frac{du_i}{dt}\left(\frac{\partial x_j}{\partial u_i}\frac{\partial y_j}{\partial u_k} - \frac{\partial y_j}{\partial u_i}\frac{\partial x_j}{\partial u_k}\right) \\ \frac{\partial F^*}{dt} &= \frac{du_i}{dt}L_{ik} \end{aligned} \tag{29}$$

ここで，

$$L_{ik} = \frac{\partial x_j}{\partial u_i}\frac{\partial y_j}{\partial u_k} - \frac{\partial y_j}{\partial u_i}\frac{\partial x_j}{\partial u_k} \quad \text{Lagrange bracket} \tag{30}$$

(29) が正準方程式となるためには，適当な変換の後に，

$$L_{ik} = \delta_{i,k-n} - \delta_{i,k+n} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \quad (\text{正準構造}) \tag{31}$$

であればよい．このとき，

$$\frac{\partial F^*}{\partial u_{n+i}} = \frac{du_i}{dt}, \quad \frac{\partial F^*}{\partial u_i} = \frac{du_{n+i}}{dt}, \quad (i = 1 - n) \tag{32}$$

実は，1 と 2 は同等な条件 (必要十分条件) である．(20) と (29) より，

$$\frac{du_i}{dt} = P_{ik} L_{hk} \frac{du_h}{dt} \tag{33}$$

よって，

$$P_{ik} L_{hk} = \delta_{ih} \tag{34}$$

いずれにしろ，場合によって P, L を使い分けることが必要である．

1.3.1 Determinant of L, P

$n = 1$ の時，

$$\det P = \begin{vmatrix} 0 & \frac{\partial(u_1,u_2)}{\partial(x,y)} \\ \frac{p(u_2,u_1)}{x,y} & 0 \end{vmatrix} = \left(\frac{\partial(u_1, u_2)}{\partial(x, y)}\right)^2 \tag{35}$$

一般に，

$$\det P = \left(\frac{\partial(u_1, \cdots, u_{2n})}{\partial(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)}\right)^2 \tag{36}$$

明らかに，

$$\det L = \frac{1}{\det P} = \left(\frac{\partial(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)}{\partial(u_1, \cdots, u_{2n})}\right)^2 \tag{37}$$

1. Jacobi-Poincaré の条件

$$x_1, y_1; x_2, y_2; \cdots; x_n, y_n \quad \rightarrow \quad u_1, u_{n+1}; u_2, u_{n+2}; \cdots; u_n, u_{2n}$$

が正準変換であるためには，

$$P \equiv y_i dx_i - u_{n+i} du_i = \text{完全微分} = d\phi(x, y) = d\phi(u) \tag{38}$$

$$\begin{aligned} P &= y_i \frac{dx_i}{du_j} du_j - u_{n+j} du_j \\ &= \left(y_i \frac{dx_i}{du_j} - u_{n+j}\right) du_j, \quad (u_{j>2n} = 0) \end{aligned} \tag{39}$$

一般に $A_j du_j$ が完全微分 (total derivation) であるためには，

$$\frac{\partial A_j}{\partial u_k} - \frac{\partial A_k}{\partial u_j} = 0 \tag{40}$$

であればよい．したがって，

$$d(A_j du_j) = 0 \tag{41}$$

であればよい．このことより，

$$\frac{\partial A_j}{\partial u_k} = \frac{\partial y_j}{\partial u_k}\frac{\partial x_i}{\partial u_j} - \frac{\partial u_{n+j}}{\partial u_k} + y_i \frac{\partial^2 x_i}{\partial u_k \partial u_j} \tag{42}$$

$$\begin{aligned} \frac{\partial A_j}{\partial u_k} - \frac{\partial A_k}{\partial u_j} &= \frac{\partial y_i}{\partial u_k}\frac{\partial x_i}{\partial u_j} - \frac{\partial y_i}{\partial u_j}\frac{\partial x_i}{\partial u_k} - \frac{\partial u_{n+j}}{\partial u_k} + \frac{\partial u_{n+k}}{\partial u_j} \\ &= L_{jk} - \delta_{j,k-n} + \delta_{j,k+n} = 0 \end{aligned} \tag{43}$$

これは L が正準構造であることを意味する．

EX 1) $x, y \rightarrow y, -x$

$$P = ydx - (-x)dy = d(xy) \qquad \text{OK}$$

EX 2) $x_1, y_1; x_2, y_2 \rightarrow x_1 + x_2, y_1; x_2, y_2 - y_1$

$$P = y_1 dx_1 + y_2 dx_2 - y_2 d(x_1 + x_2) - (y_2 - y_1)dx_2 = 0 \quad \text{OK}$$

c′) 次の 2 つが全微分であることを示してもよい．

$$P_1 = \sum xdy - \sum x'dy', \quad P_2 = \sum ydx - \sum y'dx'$$

$$P - P_1 = \sum d(xy) - \sum d(x'y') = d(\sum) = d(\sum xy - \sum x'y') \tag{44}$$

$$P_2 - P = d(\sum x'y') \tag{45}$$

d) 母関数 $S = S(x', y)$

$$x_i = \frac{\partial S}{\partial y_i}, \quad y_i' = \frac{\partial S}{\partial x_i'} \quad i = 1, \cdots n \tag{46}$$

この関係は正準変換である．

証明

$$\begin{aligned} P &= \sum ydx - \sum y'dx' \\ &= d\sum xy - \sum xdy - \sum y'dx' \\ &= d\sum xy - \left(\sum \frac{\partial S}{\partial y}dy + \sum \frac{\partial S}{\partial x'}dx'\right) \\ &= d(\sum xy - S) \end{aligned} \tag{47}$$

母関数の取り方はいろいろある.

$$S_1 = S_1(x, y') \tag{48}$$

$$y_i = -\frac{\partial S_1}{\partial x_i}, \quad x'_i = -\frac{\partial S_1}{\partial y'_i} \tag{49}$$

証明

$$P_2 = \sum y dx + \sum x' dy' = -dS$$

S の型に全くよらないのでこれは便利な方法である.

1.4　質点系の運動

$$\begin{aligned} P_1 &: x_1, x_2, x_3; m_1 = m_2 = m_3 \\ P_2 &: x_4, x_5, x_6; m_4 = m_5 = m_6 \\ & \cdots \\ P_{n/3} &: x_{n-2}, x_{n-1}, x_n; m_{n-2} = m_{n-1} = m_n \end{aligned}$$

運動方程式[1]

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i} \quad (i = 1, \cdots n) \tag{50}$$

$$U = U(x_1, x_2, \cdots, x_n) \tag{51}$$

共変量

$$y_i = m_i \frac{dx_i}{dt} = m_i \dot{x}_i \quad (i = 1, \cdots, n) \tag{52}$$

とすると (50) は,

$$\frac{dy_i}{dt} = \frac{\partial U}{\partial x_i} \tag{53}$$

さらに,

$$T = \frac{1}{2} \sum m_i \dot{x}_i^2 = \frac{1}{2} \sum \frac{y_i^2}{m_i} \tag{54}$$

とおくと,

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i = y_i \tag{55}$$

$$\frac{\partial T}{\partial y_i} = \frac{y_i}{m_i} = \dot{x}_i = \frac{dx_i}{dt} \tag{56}$$

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[1] U はポテンシャルの符号を逆にしたもの.

$$\left[\begin{array}{l} \frac{dx_i}{dt} = \frac{\partial(T-U)}{\partial y_i} \\ \frac{dy_i}{dt} = -\frac{\partial(T-U)}{\partial x_i} \end{array}\right. \qquad (i = 1, \cdots n) \tag{57}$$

$$T - U = \text{const.} \quad \text{エネルギー積分}$$

ただし，回転座標系では全エネルギーを表さない．

任意の座標系へ移そう．

1) 正準変換 $x, y \to q(x), p(x, y)$ ((c) を見よ．)

条件は c) より，

$$P = y_i dx_i - p_j dq_j \tag{58}$$

が全微分．さて，

$$dx_i = \frac{\partial x_i}{\partial q_j} dq_j, \quad P = \left(y_i \frac{\partial x_i}{\partial q_j} - p_j \right) dq_j \tag{59}$$

つまり，

$$p_j = y_i \frac{\partial x_i}{\partial q_j} \tag{60}$$

と与えれば十分である[2]．さて，

$$\dot{x}_i = \frac{\partial x_i}{\partial q_j} \dot{q}_j \tag{61}$$

より，

$$T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 = \frac{1}{2} \sum_i m_i \left(\frac{\partial x_i}{\partial q_j} \dot{q}_j \right)^2 \tag{62}$$

よって，

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_i} &= \sum m \left(\frac{\partial x}{\partial q_j} \dot{q}_j \right) \frac{\partial x}{\partial q_i} = \sum m \dot{x} \frac{\partial x}{\partial q_i} = \sum y \frac{\partial x}{\partial q_i} \\ p_i &= \frac{\partial T}{\partial \dot{q}_i} \end{aligned} \tag{63}$$

(63) を用いて $\dot{q} \to p$ で表す．

$$T(q, \dot{q}) \to T(p, q), \quad U(x) \to U(q)$$

$$\left[\begin{array}{l} \frac{dq_i}{dt} = \frac{\partial(T-U)}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial(T-U)}{\partial q_i} \end{array}\right. \qquad (i = 1, \cdots, n) \tag{64}$$

$$T(p, q) - U(q) = \text{const.}$$

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[2]

$$y_i \frac{\partial x_i}{\partial q_j} = y \frac{\partial x}{\partial q_j}$$

等と書く．

2) 束縛運動

$N/3$ 個の質点の束縛運動．このとき，$x_1, \cdots, x_N$ は $n(n < N)$ 個の一般化座標で記述される．

$$x_i = \phi_i(q_1, \cdots, q_n) = \psi_i(q_1, \cdots, x_N) \quad (i = 1, \cdots, N) \tag{65}$$

または，

$$\psi(q_1, \cdots, q_n, \underbrace{0, \cdots, 0}_{N-n\text{ 個}}) = \phi_i(q_1, \cdots, q_n) \tag{66}$$

と考えても良い (束縛条件のため $N - n$ 個の 0 となる)．

$$\begin{aligned} T &= \frac{1}{2}\sum m\left(\frac{\partial x}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial x}{\partial q_N}\dot{q}_N\right)^2 \\ &= \frac{1}{2}\sum m\left(\frac{\partial x}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial x}{\partial q_n}\dot{q}_n\right)^2 \end{aligned} \tag{67}$$

運動方程式は，

$$\left[\begin{array}{l} \frac{dq_i}{dt} = \frac{\partial(T-U-U')}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial(T-U-U')}{\partial q_i} \end{array}\right. \qquad (i = 1, \cdots, N) \tag{68}$$

ここで，U' は束縛運動の Potential．

束縛 $q_{n+n} = q_{n+2} = \cdots = q_N = 0$ は $U'(q_1, \cdots, q_N)$ から導かれる．

滑らかな束縛 (束縛に逆らわない変位に対して仕事をしない) では，

$$\begin{aligned} \delta U' &= \frac{\partial U'}{\partial q_1}\delta q_1 + \cdots + \frac{\partial U'}{\partial q_n}\delta q_n = 0 \\ &\to \frac{\partial U'}{\partial q_1} = \frac{\partial U'}{\partial q_2} = \cdots = \frac{\partial U'}{\partial q_n} = 0 \end{aligned} \tag{69}$$

このとき (50) は，

$$\left[\begin{array}{l} \frac{dq_i}{dt} = \frac{\partial(T-U)}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial(T-U)}{\partial q_i} \end{array}\right. \qquad (i = 1, \cdots, n) \tag{70}$$

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \quad (i = 1, \cdots, n)$$

注：

$p_i = \frac{\partial T}{\partial \dot{q}_i}$ から果して $\dot{q} \to p$ で表せるか？

上の式は連立１次．

$$\sum m\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_i}\dot{q}_1 + \cdots + \sum m\frac{\partial x}{\partial q_n}\frac{\partial x}{\partial q_n}\dot{q}_n = p_i \quad (i = 1, \cdots, n) \tag{71}$$

$\dot{q}_j$ の係数の作る行列式を Δ とする．$\Delta = 0$ なら，

$$\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 0 \quad \text{for some} \quad \dot{q}_j \neq 0 \tag{72}$$

すると，Euler より T は $\dot{q}_j$ の 2 次式だから，

$$\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T = 0 \tag{73}$$

すなわち，

$$\frac{\partial x_i}{\partial \dot{q}_j}\dot{q}_j = 0 \quad \text{for some} \dot{q}_j \neq 0 \quad (i = 1, \cdots, N) \tag{74}$$

つまり $N \times n$ の行列

$$\mathrm{rank}\left(\frac{\partial x_i}{\partial \dot{q}_j}\right) < n, \quad i = 1, \cdots, N, j = 1, \cdots, n \tag{75}$$

これより，x_i は n 個以下の新しい一般化座標で表されることになる．
ex. 平面運動；

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{76}$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \tag{77}$$

$$p_\theta = \frac{\partial T}{\partial \dot{r}} = mr^2\dot{\theta} \tag{78}$$

$$\frac{\partial T}{\partial \dot{q}_j} = p_i \quad \rightarrow \quad \begin{cases} m\dot{r} + 0 \cdot \dot{\theta} = p_r \\ 0 \cdot r + mr^2\dot{\theta} = p_\theta \\ \Delta = m^2 r^2 \end{cases} \tag{79}$$

$r = 0$ で $\Delta = 0$，このとき p_r, p_θ で r, θ は表せない (これは変換の特異点)．

1.5 ハミルトニアン F が t を explicit に含む場合

$$\left[\begin{array}{l} \frac{dx_i}{dt} = \frac{\partial}{\partial y_i}F(x, y, t) \\ \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i}F(x, y, t) \end{array}\right. \tag{80}$$

これは，t を explicit に含まない場合に帰着できる．x_{n+1}, y_{n+1} を新たに導入し

$$F' = F(x, y, x_{n+1}) + y_{n+1}$$

とおく．

$$\frac{dx_i}{dt} = \frac{\partial F'}{\partial y_i} = \frac{\partial}{\partial y_i}F(x, y, x_{n+1}), \quad \frac{dy_i}{dt} = -\frac{\partial F'}{\partial x_i} = -\frac{\partial}{\partial x_i}F(x, y, x_{n+1}) \tag{81}$$

$$\frac{dx_{n+1}}{dt} = \frac{\partial F'}{\partial y_{n+1}} = 1 \quad \left(\frac{dt}{dt} = 1\right) \tag{82}$$

$$\frac{dy_{n+1}}{dt} = -\frac{\partial F'}{\partial x_{n+1}} = -\frac{\partial}{\partial x_{n+1}}F(x, y, x_{n+1}) = -\frac{\partial F}{\partial t} \tag{83}$$

y_{n+1} については意味がないので，どうでもよい．
積分

$$F' = \mathrm{const.} \tag{84}$$

(81) + (82) $\cdots$ $2n+2$ 階 $\Rightarrow$ $2n$ 階の系と 1 つの求積となる．

1.5.1 正準変換の条件

d) の正準変換

$$x, x_{n+1}; y, y_{n+1} \quad \Rightarrow \quad q, q_{n+1}; p, p_{n+1}$$

母関数：

$$\begin{aligned} S &= S(q, q_{n+1}; y, y_{n+1}) \\ &= S^*(q, q_{n+1}, y) + y_{n+1} q_{n+1} \end{aligned} \tag{85}$$

と仮定する．変換：

$$\left.\begin{array}{l} p_i = \frac{\partial S}{\partial q_i} = \frac{\partial S^*}{\partial q_i}, \quad x_i = \frac{\partial S}{\partial y_i} = \frac{\partial S^*}{\partial y_i} \\ p_{n+1} = \frac{\partial S}{\partial q_{n+1}} = \frac{\partial S^*}{\partial q_{n+1}} + y_{n+1}, \quad x_{n+1} = \frac{\partial S}{\partial y_{n+1}} = q_{n+1} \end{array}\right\} \quad (1 = 1, \cdots, n) \tag{86}$$

ハミルトニアン：

$$\begin{aligned} F^* &= F(x(q,p), y(q,p), q_{n+1}) \\ &\quad + p_{n+1} - \frac{\partial}{\partial q_{n+1}} S^*(q, q_{n+1}, y(q,p)) \end{aligned} \tag{87}$$

運動方程式：

$$\frac{dq_i}{dt} = \frac{\partial F^*}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial F^*}{\partial q_i}, \quad \frac{dq_{n+1}}{dt} = \frac{\partial F^*}{\partial p_{n+1}} = 1, \quad \frac{dp_{n+1}}{dt} = -\frac{\partial F^*}{\partial q_{n+1}} \tag{88}$$

書き直すと，

$$\frac{dq_i}{dt} = \frac{\partial}{\partial p_i}\left(F - \frac{\partial S^*}{\partial q_{n+1}}\right) \tag{89}$$

$$\frac{dp_i}{dt} = -\frac{\partial}{\partial q_i}\left(F - \frac{\partial S^*}{\partial q_{n+1}}\right) \tag{90}$$

$$q_{n+1} = t \tag{91}$$

$$F - \frac{\partial S^*}{\partial q_{n+1}} + p_{n+1} = \mathrm{const.} \tag{92}$$

より p_{n+1} はどうでもよい．このようにハミルトニアンは $\frac{\partial S^*}{\partial t}$ の分だけ異なって来る．

1.6　正準変換の一般化

$$x, y : x', y'$$

変換が次の条件を満足するとする．

$$\sum y dx - \sum y' dx' = d\varphi + \psi dt \tag{93}$$

φ, ψ： x, y, t or x', y', t の関数．そして，$x, y(x', y')$ は第 2 の parameter C にも依存すると仮定する．
Lagrange bracket $[t, C]$ を $x, y(x', y')$ で評価する．(93) より，

$$\frac{\partial}{\partial C} \quad \times \quad \left[\sum y \frac{\partial x}{\partial t} - \sum y' \frac{\partial x'}{\partial t} - \psi = \frac{\partial \varphi}{\partial t} \right] \tag{94}$$

$$\frac{\partial}{\partial t} \quad \times \quad \left[\sum y \frac{\partial x}{\partial C} - \sum y' \frac{\partial x'}{\partial C} = \frac{\partial \varphi}{\partial C} \right] \tag{95}$$

2 式を引き算して，

$$\sum \left(\frac{\partial y}{\partial C} \frac{\partial x}{\partial t} - \frac{\partial y}{\partial t} \frac{\partial x}{\partial C} \right) - \frac{\partial \psi}{\partial C} = \sum \left(\frac{\partial y'}{\partial C} \frac{\partial x'}{\partial t} - \frac{\partial y'}{\partial t} \frac{\partial x'}{\partial C} \right) \tag{96}$$

つまり，

$$[t, C]_{x,y} - \frac{\partial \psi}{\partial C} = [t, C]_{x',y'} \tag{97}$$

今，

$$\frac{dx_i}{dt} = Y_i(x, y, t), \quad \frac{dy_i}{dt} = -X_i(x, y, t) \quad (i = 1, \cdots, n) \tag{98}$$

を仮定する．これに変換 (93) を施す．仮に (98) を解けば x, y は t と $2n$ 個の積分定数 $C_1, \cdots, C_{2n}$ で表される．今，C を δC 変えたときの変分を $\delta x, \delta y, \delta \psi$ とする．
さて，

$$A \equiv \sum X \delta x + \sum Y \delta y - \delta \psi \tag{99}$$

とおくと，δC の係数は，

$$A_C = \sum X \frac{\partial x}{\partial C} + \sum Y \frac{\partial y}{\partial C} - \frac{\partial \psi}{\partial C} \tag{100}$$

$$A_C = [t, C]_{x,y} - \frac{\partial \psi}{\partial C} \tag{101}$$

ところで A を x', y' で書くと，

$$A = \sum X' \delta x' + \sum Y' \delta y' \tag{102}$$

勿論 A_C は共通でなければならないから,

$$A_C = \sum X' \frac{\partial x'}{\partial C} + \sum Y' \frac{\partial y'}{\partial C} \tag{103}$$

したかって,

$$[t, C]_{x,y} - \frac{\partial \psi}{\partial C} = \sum X' \frac{\partial x'}{\partial C} + \sum Y' \frac{\partial y'}{\partial C} \tag{104}$$

$$[t, C]_{x',y'} = \sum \left(-\frac{\partial y'}{\partial t} \frac{\partial x'}{\partial C} + \frac{\partial x'}{\partial t} \frac{\partial y'}{\partial t} \right) \tag{105}$$

よって,

$$\sum \frac{\partial x'}{\partial C} \left(X + \frac{\partial y'}{\partial t} \right) + \sum \frac{\partial y'}{\partial C} \left(Y - \frac{\partial x'}{\partial t} \right) = 0 \tag{106}$$

これは, $2n$ 個の未知数を持つ $2n$ 個の連立 1 次 (斉次) 方程式.
よって,

$$\Delta = \frac{\partial(x'_1, \cdots, x'_n; y'_1, \cdots, y'_n)}{\partial(C_1, \cdots, C_{2n})} \neq 0 \tag{107}$$

ならば,

$$\frac{dx'_i}{dt} = Y'_i, \quad \frac{dy'_i}{dt} = -X'_i \quad (i = 1, \cdots, n) \tag{108}$$

$\Delta \neq 0$ は $C_1, \cdots, C_{2n}$ が独立なら成り立つ. 特に,

$$X_i = \frac{\partial F}{\partial x_i}, \quad Y_i = \frac{\partial F}{\partial y_i} \tag{109}$$

なら,

$$A = \sum \frac{\partial F}{\partial x} \delta x + \sum \frac{\partial F}{\partial y} \delta y - \delta \psi = \delta(F - \psi) \tag{110}$$

よって, F, ψ を x', y' で表したものを使って,

$$A = \sum \frac{\partial(F - \psi)}{\partial x'} \delta x' + \sum \frac{\partial(F - \psi)}{\partial y'} \delta y' \tag{111}$$

よって,

$$X'_i = \frac{\partial(F - \psi)}{\partial x'_i}, \quad Y'_i = \frac{\partial(F - \psi)}{\partial y'_i} \tag{112}$$

これは確かに正準変換,

$$\frac{dx'_i}{dt} = \frac{\partial(F - \psi)}{\partial y'_i}, \quad \frac{dy'_i}{dt} = -\frac{\partial(F - \psi)}{\partial x'_i} \quad (i = 1, \cdots, n) \tag{113}$$

これは,

$$\psi = \frac{\partial S^*}{\partial q_{n+1}} = \frac{\partial S^*}{\partial t} \tag{114}$$

に対応していることが分かる. これは正準変換の一般化と呼ぶにふさわしい.

1.7　質点系への応用

$$m_i \frac{d^2 x_i}{dt^2} = X_i \left(x, \frac{dx}{dt}, t \right) \quad (i = 1, \cdots, n) \tag{115}$$

$$y_i = m \frac{dx_i}{dt} \tag{116}$$

運動方程式：

$$\frac{dx_i}{dt} = \frac{y_i}{m_i}, \quad \frac{dy_i}{dt} = X_i = -(-X_i) \quad (i = 1, \cdots, n) \tag{117}$$

一般化座標：

$$x_i = x_i(q_1, \cdots, q_n, t) \quad (i = 1. \cdots, n) \tag{118}$$

q_i に共役な運動量 p_i．p は $\sum y dx - \sum p dq = d\varphi + \psi dt$ (*) を満足するように取る．すると，

$$dx = \frac{\partial x}{\partial q_j} dx_j + \frac{\partial x}{\partial t} dt \tag{119}$$

(*) の右辺，

$$y_j dx^j + p_i dq^i = \left(y_j \frac{\partial x^j}{\partial q^i} - p_i \right) dq^i + y_j \frac{\partial x^j}{\partial t} dt \tag{120}$$

よって，

$$p_i = y_j \frac{\partial x^j}{\partial q^i}, \quad \psi = y_j \frac{\partial x^j}{\partial t} \tag{121}$$

で充分である．

$$\begin{aligned} T &= \frac{1}{2} m_j (\dot{x}^j)^2 = \frac{1}{2} m_j \left(\frac{\partial x^j}{\partial q^k} \dot{q}^k + \frac{\partial x^j}{\partial t} \right)^2 \\ &= \underbrace{T_2}_{\dot{q}\text{の 2 次}} + \underbrace{T_1}_{\dot{q}\text{の 1 次}} + \underbrace{T_0}_{\dot{q}\text{の 0 次}} \end{aligned} \tag{122}$$

このとき，

$$\frac{\partial T}{\partial \dot{q}_i} = m_j \dot{x}^j \frac{\partial x^j}{\partial q_i} = y_j \frac{\partial x^j}{\partial q_i} = p_i \tag{123}$$

or

$$\frac{\partial T_2}{\partial \dot{q}_i} = -\frac{\partial T_1}{\partial \dot{q}_i} + p_i \quad (i = 1, \cdots, n) \tag{124}$$

これは $\dot{q}$ の連立 1 次方程式．これを解けば $\dot{q}_j = \dot{q}_j(p, q, t)$．
さて ψ は，

$$T_1 = \sum m \left(\frac{\partial x}{\partial q^k} \dot{q}^k \right) \left(\frac{\partial x}{\partial t} \right), \quad T_0 = \frac{1}{2} \sum m \left(\frac{\partial x}{\partial t} \right)^2 \tag{125}$$

これより，

$$\begin{aligned} T_1 + 2T_0 &= \sum m \left(\frac{\partial x}{\partial q^k} \dot{q}^k + \frac{\partial x}{\partial t} \right) \left(\frac{\partial x}{\partial t} \right) \\ &= \sum m \dot{x} \frac{\partial x}{\partial t} = \sum y \frac{\partial x}{\partial t} = \psi \end{aligned} \tag{126}$$

すると，

$$A = -\sum X \delta x + \sum \frac{y}{m} \delta y - \delta(T_1 + 2T_0) \tag{127}$$

イ) $\sum X\delta x$: 変位 δx による仕事.

q に対する一般化力を Q とすれば $\sum X\delta x = \sum Q dq$.

ロ)

$$T = \frac{1}{2}\sum m\dot{x}^2 = \frac{1}{2}\frac{y^2}{m} \to \delta T = \sum \frac{y}{m}\delta y \tag{128}$$

よって，

$$\begin{aligned} A &= -\sum Q\delta q + \delta(T - T_1 - 2T_0) \\ &= -\sum Q\delta q + \delta(T_2 - T_0) \\ &= -\sum Q\delta q + \sum \frac{\partial(T_2 - T_0)}{\partial q}\delta q + \sum \frac{\partial(T_2 - T_0)}{\partial p}\delta p \end{aligned} \tag{129}$$

新しい運動方程式：

$$\frac{dq_i}{dt} = \frac{\partial(T_2 - T_0)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial(T_2 - T_0)}{\partial q_i} + Q_i \quad (i = 1, \cdots, n) \tag{130}$$

特に一般化力が保存力なら，

$$\sum Q\delta q = \delta U(q, t) \tag{131}$$

このとき，

$$\frac{dq_i}{dt} = \frac{\partial(T_2 - T_0 - U)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial(T_2 - T_0 - U)}{\partial q_i} \quad (i = 1, \cdots, n) \tag{132}$$

さらに，$x = x(q)$，t-implicit なら $T_1 = T_0 = 0$.

ハミルトニアン：

$$T_2 - T_0 - U = T - U \tag{133}$$

全エネルギーとなる．また (132) で $T_2 - T_0 - U$ が t-implicit なら，

$$T_2 - T_0 - U = \text{const.}, \quad (\text{積分}) \tag{134}$$

となる．一般にはハミルトニアンが全エネルギーになるとは限らない[3]．

[3] 例えば 3 体問題の Jacobi 積分.

1.8 種々の正準変換

1.8.1 $x, y, z, p_x, p_y, p_z \rightarrow \xi, \eta, \zeta, p_\xi, p_\eta, p_\zeta$ (回転系)

母関数

$$\begin{aligned} S &= S(\xi, \eta, \zeta, p_x, p_y, p_z) \\ &= p_x(\xi\cos\theta - \eta\sin\theta) + p_y(\xi\sin\theta + \eta\cos\theta) + p_z\zeta \end{aligned} \tag{135}$$

とすればよい[4]．実際，変換，

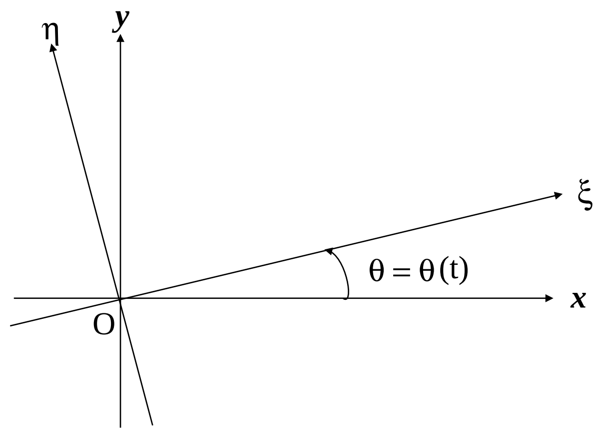

Figure 1: $x, y, z, p_x, p_y, p_z \rightarrow \xi, \eta, \zeta, p_\xi, p_\eta, p_\zeta$

$$x = \frac{\partial S}{\partial p_x} = \xi\cos\theta - \eta\sin\theta \tag{136}$$

$$y = \frac{\partial S}{\partial p_y} = \xi\sin\theta + \eta\cos\theta \tag{137}$$

$$z = \frac{\partial S}{\partial p_z} = \zeta \tag{138}$$

$$p_\xi = \frac{\partial S}{\partial \xi} = p_x\cos\theta + p_y\sin\theta \tag{139}$$

$$p_\eta = \frac{\partial S}{\partial \eta} = -p_x\sin\theta + p_y\cos\theta \tag{140}$$

$$p_\zeta = \frac{\partial S}{\partial \zeta} = p_z \tag{141}$$

4

$$\begin{aligned} x &= \xi\cos\theta - \eta\sin\theta \\ y &= \xi\sin\theta + \eta\cos\theta \\ z &= \zeta \end{aligned}$$

を解いて，

$$\begin{cases} p_x = p_\xi \cos\theta - p_\eta \sin\theta \\ p_y = p_\xi \sin\theta + p_\eta \cos\theta \\ p_z = p_\zeta \end{cases} \tag{142}$$

$$\begin{aligned} F^* &= F - \frac{\partial S}{\partial t} = F - \frac{\partial S}{\partial \theta}\frac{d\theta}{dt} \\ &= F + [p_x(\xi\sin\theta + \eta\cos\theta) - p_y(\xi\cos\theta - \eta\sin\theta)]\,\dot\theta \\ &= F + (-\xi p_\eta + \eta p_\xi)\dot\theta \end{aligned} \tag{143}$$

$$\begin{aligned} F &= \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - U \\ &= \frac{1}{2}(p_\xi^2 + p_\eta^2 + p_\zeta^2) - U \end{aligned} \tag{144}$$

よって，

$$F^* = \frac{1}{2}(p_\xi^2 + p_\eta^2 + p_\zeta) + 2\dot\theta(\eta p_\xi - \xi p_\eta) - U(\xi, \eta, \zeta, t) \tag{145}$$

新運動方程式

$$\frac{d\xi}{dt} = \frac{\partial F^*}{\partial p_\xi} = p_\xi + \dot\theta\eta \tag{146}$$

$$\frac{d\eta}{dt} = \frac{\partial F^*}{\partial p_\eta} = p_\eta - \dot\theta\xi \tag{147}$$

$$\frac{d\zeta}{dt} = \frac{\partial F^*}{\partial p_\zeta} = p_\zeta \tag{148}$$

$$\frac{dp_\xi}{dt} = -\frac{\partial F^*}{\partial \xi} = \dot\theta p_\eta + \frac{\partial U}{\partial \xi} \tag{149}$$

$$\frac{dp_\eta}{dt} = -\frac{\partial F^*}{\partial \eta} = -\dot\theta p_\xi + \frac{\partial U}{\partial \eta} \tag{150}$$

$$\frac{d\zeta}{dt} = -\frac{\partial F^*}{\partial \zeta} = \frac{\partial U}{\partial \zeta} \tag{151}$$

これより $\dot p$ を消去して，

$$\frac{d^2\xi}{dt^2} - \underline{2\dot\theta\frac{d\eta}{dt}} - \ddot\theta^2\eta = \frac{\partial\Omega}{\partial\xi} \tag{152}$$

$$\frac{d^2\eta}{dt^2} + \underline{2\dot\theta\frac{d\xi}{dt}} + \ddot\theta^2\xi = \frac{\partial\Omega}{\partial\eta} \tag{153}$$

$$\frac{d^2\zeta}{dt^2} = \frac{\partial\Omega}{\partial\zeta} \tag{154}$$

$$\Omega = U + \frac{\dot\theta^2}{2}(\xi^2 + \eta^2) \quad ; \quad \text{Synodic Potential (遠心力ポテンシャル)} \tag{155}$$

ここで，アンダーラインの項はコリオリ力である[5]．

[5] この運動方程式は別な方法で求めたほうが早い．

1.8.2 Jacobi 座標

慣性系 ここで

	質量	座標
月	m_1, m_2, m_3	x_1, x_2, x_3
太陽	m_4, m_5, m_6	x_4, x_5, x_6
地球	m_7, m_8, m_9	$x_7.x_8, x_9$

$$y_i = m_i \frac{dx_i}{dt} \tag{156}$$

運動方程式

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \tag{157}$$

ハミルトニアン

$$F = \sum \frac{y_i^2}{2m_i} - U(x) \tag{158}$$

ヤコビ座標

$$\begin{cases} \xi_i = x_i - x_{i-6}, \quad (\text{月}) \\ \xi_{i+3} = x_{i+3} - \frac{m_1 x_i + m_7 x_{i+6}}{m_1 + m_7}, \quad (\text{太陽}) \qquad (i = 1, 2, 3) \\ \xi_{i+6} = 0, \quad (\text{地球}) \end{cases} \tag{159}$$

母関数

$$\begin{aligned} S &= S(x, \eta) \\ &= -\sum_{i=1}^{3} \left[\eta_i (x_i - x_{i+6}) + \eta_{i+3} \left(x_{i+3} - \frac{m_1 x_i + m_7 x_{i+6}}{m_1 + m_7} \right) \right] \end{aligned} \tag{160}$$

とおけばよい．すると，

$$\xi_i = -\frac{\partial S}{\partial \eta_i} = x_i - x_{i+6} \tag{161}$$

$$x_{i+3} = -\frac{\partial S}{\partial \eta_{i+3}} = x_{i+3} - \frac{m_1 x_i + m_7 x_{i+6}}{m_1 + m_7} \tag{162}$$

$$y_i = -\frac{\partial S}{\partial x_i} = \eta_i - \frac{m_1}{m_1 + m_7} \eta_{i+3} \tag{163}$$

$$y_{i+3} = \frac{\partial S}{\partial x_{i+3}} = \eta_{i+3} \tag{164}$$

$$y_{i+6} = -\frac{\partial S}{\partial x_{i+6}} = -\eta_i - \frac{m_7}{m_1 + m_7} \eta_{i+6} \tag{165}$$

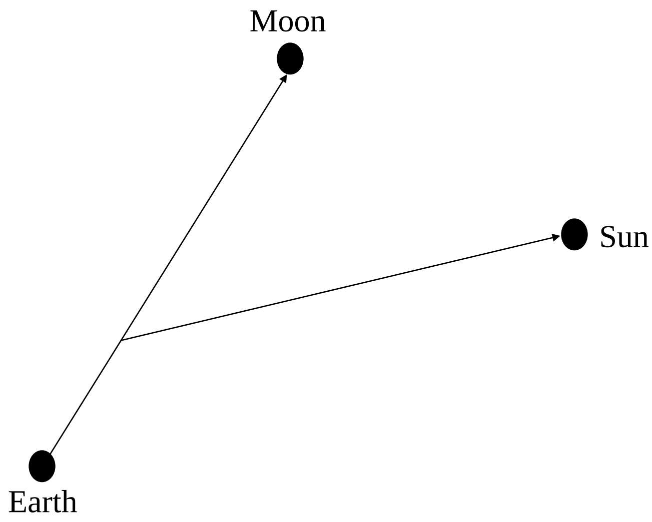

Figure 2: ヤコビ座標

新運動方程式,

$$\frac{d\xi_i}{dt} = \frac{\partial F^*}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial F^*}{\partial \xi_i}, \quad i = 1, \cdots, 6 \tag{166}$$

$$\begin{aligned} F^* &= F(x, y) \\ &= \frac{1}{2m_1}\sum_{i=1}^{3}\left(\eta_i - \frac{m_1}{m_1+m_7}\eta_{i+3}\right)^2 + \frac{1}{2m_4}\sum_{i=1}^{3}\eta_{i+3}^2 \\ &\quad + \frac{1}{2m_7}\sum_{i=1}^{3}\left(\eta_i + \frac{m_7}{m_1+m_7}\eta_{i+3}\right)^2 - U(\xi) \\ &= \frac{1}{2m_1}\left[\sum \eta_i^2 + \left(\frac{m_1}{m_1+m_7}\right)^2 \sum \eta_{i+3}^2\right] \\ &\quad + \frac{1}{2m_4}\sum \eta_{i+3}^2 + \frac{1}{2m_7}\left[+\sum_{i}^{2}\left(\frac{m_7}{m_1+m_7}\right)^2 \sum \eta_{i+3}\right] - U(\xi) \end{aligned} \tag{167}$$

ここで,

$$\sum \eta_i^2 \quad : \quad \frac{1}{2m_1} + \frac{1}{2m_7} = \frac{1}{2m_1'} \to m_1' = \frac{m_1 m_7}{m_1 + m_7} \tag{168}$$

$$\sum \eta_{i+3}^2 \quad : \quad \frac{1}{2(m_1+m_7)} + \frac{1}{2m_4} = \frac{1}{2m_4'} \to m_4' = \frac{m_4(m_1+m_7)}{m_4+m_1+m_7} \tag{169}$$

すると,

$$F^*(\xi, \eta) = \frac{1}{2m_1'}\sum \eta_i^2 + \frac{1}{2m_4'}\eta_{i+3}^2 - U(\xi) \tag{170}$$

1.8.3 Poisson 変換

$$\xi = k\sqrt{2x}\cos y, \quad \eta = k^{-1}\sqrt{2x}\sin y, \quad k = \text{constant} \tag{171}$$

母関数を，

$$S = S(\xi, y) = \frac{1}{2k^2}\xi^2 \tan y \tag{172}$$

とすればよい．

$$x = \frac{\partial S}{\partial y} = \frac{\xi^2}{2k^2}\sec^2 y \to \xi = k\sqrt{2x}\cos y \tag{173}$$

$$\eta = \frac{\partial S}{\partial \xi} = \frac{1}{k^2}\xi \tan y \to \eta = \frac{1}{k}\sqrt{2x}\sin y \tag{174}$$

1.8.4 Delauney 変換

$$\begin{cases} \frac{dp_r}{dt} = \frac{\partial F}{\partial r}, & \frac{dp_\theta}{dt} = \frac{\partial F}{\partial \theta}, & \frac{dp_\varphi}{dt} = \frac{\partial F}{\partial \varphi} \\ \frac{dr}{dt} = -\frac{\partial F}{\partial p_r}, & \frac{d\theta}{dt} = -\frac{\partial F}{\partial p_\theta}, & \frac{d\varphi}{dt} = -\frac{\partial F}{\partial p_\varphi} \end{cases} \tag{175}$$

$$F = -\frac{1}{2}\left(P_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + \frac{\mu}{r} + R(r, \theta, \varphi, t) \tag{176}$$

$R = 0$ の解 → Kepler 運動

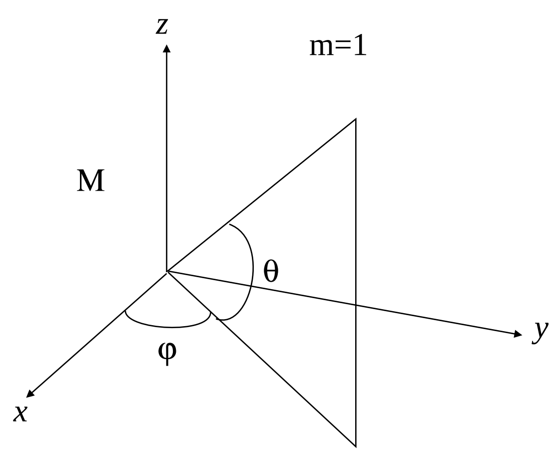

Figure 3: 極座標

$$a, e, i, \Omega, \omega, T, \quad \text{Kepler 要素}$$

Delauney 要素 (変数)

$$\begin{cases} L = \sqrt{\mu a}, \quad G = \sqrt{\mu a(1-e^2)}, \quad H = \sqrt{\mu a(1-e^2)}\cos i \\ \ell = n(t-T), \quad g = \omega, \quad h = \Omega \end{cases} \tag{177}$$

エネルギー積分，

$$\frac{\mu}{2a} = F_{\text{Kepler}} \tag{178}$$

これより，

$$\frac{\mu^2}{2L^2} = -\frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + \frac{\mu}{r} \tag{179}$$

角運動量保存

$$\sqrt{\mu a(1-e^2)} = M \tag{180}$$

より，

$$G = \left(p_\theta^2 + \frac{p_\varphi^2}{\cos^2\theta}\right)^{\frac{1}{2}} \tag{181}$$

さらに，

$$H = p_\varphi \tag{182}$$

これらを p_r, p_θ, p_φ について解いて，

$$\begin{cases} p_r = \pm\left(-\frac{G^2}{r^2} + \frac{2\mu}{r} - \frac{\mu^2}{L^2}\right)^{\frac{1}{2}} \\ p_\theta = \pm\left(G^2 - \frac{H^2}{\cos^2\theta}\right)^{\frac{1}{2}} \\ p_\varphi = H \end{cases} \tag{183}$$

母関数，

$$\begin{aligned} S &= S(L, G, H; r, \theta, \varphi) \\ &= \int p_r dr + \int p_\theta d\theta + \int p_\varphi d\varphi \\ &= \int_{r_m(L,G)} \left(-G^2 + 2\mu r - \frac{\mu^2}{L^2} r^2\right)^{\frac{1}{2}} \frac{dr}{r} \\ &\quad + \int_0 \left(G^2 - \frac{H^2}{\cos^2\theta}\right)^{\frac{1}{2}} d\theta + H\varphi \end{aligned} \tag{184}$$

さて，

$$-G^2 + 2\mu r - \frac{\mu^2}{L^2} r^2 = \frac{\mu^2}{L^2}(r_M - r)(r - r_m) \tag{185}$$

$$r_M, r_m = \frac{L^2}{\mu}\left(1 \pm \sqrt{1 - \frac{G^2}{L^2}}\right) = a(1 \pm e) \tag{186}$$

i)

$$\begin{aligned} \ell = \frac{\partial S}{\partial L} &= \frac{\mu^2}{L^2}\int_{r_m}\left(-G^2 + 2\mu r - \frac{\mu^2}{L^2}r^2\right)^{-\frac{1}{2}} \frac{dr}{r} \\ &= \frac{\mu}{L^2}\int \frac{r dr}{\sqrt{(r_M - r)(r - r_m)}} \\ &\quad \left(r_m \text{は integrand} = 0 \text{ の根だから} \frac{\partial r_m}{\partial L}\text{Integrand}|_{r=r_m} = 0\right) \end{aligned} \tag{187}$$

$r = a(1 - e\cos u)$ とおくと，

$$dr = ae\sin u du, \quad \sqrt{(r_M - r)(r - r_m)} = ae\sin u \tag{188}$$

$$\ell = \frac{1}{a}\int_0 a(1 - e\cos u)du = u - e\sin u, \quad \text{(Kepler Eq.)} \tag{189}$$

この ℓ を平均近点角という．

ii)

$$\begin{aligned} h = \frac{\partial S}{\partial H} &= \varphi - H\int_0 \left(G^2 - H^2\sec^2\theta\right)^{-\frac{1}{2}}\sec^2\theta d\theta \\ &= \varphi - \int_0 \left(\frac{G^2 - H^2}{H^2} - \tan^2\theta\right)^{-\frac{1}{2}} d(\tan\theta) \\ &= \varphi - \sin^{-1}\left(\frac{\tan\theta}{\tan i}\right), \quad \left(\tan^2 i equiv \frac{G^2 - H^2}{H^2}\right) \end{aligned} \tag{190}$$

よって，

$$\sin(\varphi - h) = \frac{\tan\theta}{\tan i} \rightarrow h = \Omega \quad (H = \Omega\cos i) \tag{191}$$

iii)

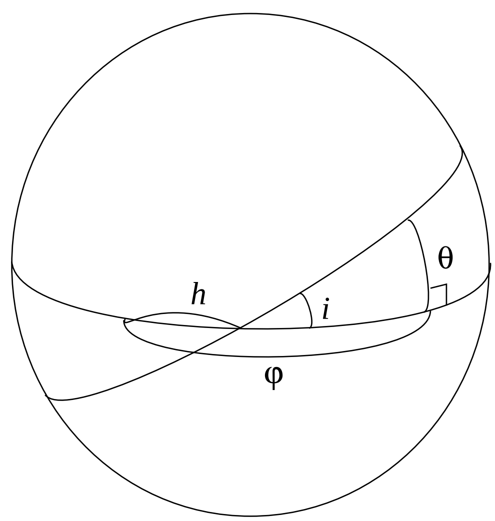

Figure 4: h, φ, i, θ

$$\begin{aligned} g = \frac{\partial S}{\partial G} &= \int_0 \left(\tan^2 i - \tan^2\theta\right)^{-\frac{1}{2}} d(\tan\theta) \\ & -\frac{LG}{\mu}\int_{r_m} \frac{dr}{r\sqrt{(r_M - r)(r - r_m)}} \end{aligned}$$

$$
\begin{aligned}
&= \sin^{-1}\left(\frac{\sin\theta}{\sin i}\right) - \sqrt{1-e^2}\int \frac{du}{1-e\cos u} \\
&= \mathrm{NP} - 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) \\
&= \mathrm{NP} - 2\tan^{-1}\left(\tan\frac{f}{2}\right) = \mathrm{NP} - f \to g = \omega \qquad (192)
\end{aligned}
$$

$$
\mathrm{NP} - f \longrightarrow g = \omega \qquad (193)
$$

ここで，A は近日点．S には ℓ を通してしか t は現れない．つまり t-implicit $\Rightarrow F^* = F$．

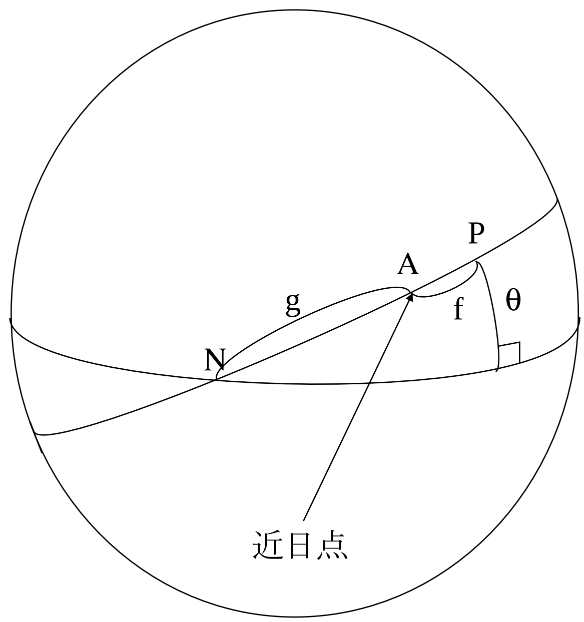

Figure 5: NP, g, f, θ

$$
F^* = \frac{\mu^2}{2L^2} + R(L, G, H; \ell, g, h; t) \qquad (194)
$$

新運動方程式は，

$$
\frac{d}{dt}(L, G, H) = \frac{\partial F}{\partial(\ell, g, h)}, \quad \frac{d}{dt}(\ell, g, h) = -\frac{\partial F}{\partial(L, G, H)} \qquad (195)
$$

$$
F = \frac{\mu^2}{2L^2} + R(L, G, H; \ell, g, h : t) \qquad (196)
$$

これは楕円軌道であることを考えている．

1.8.5　変形 Delauney 変数

i, e は普通小さいので i の order，e の order の変数で表したい．
a)[6]

$$\begin{cases} x_1 = L \\ x_2 = L - G \\ x_3 = G - H \end{cases}, \quad \begin{cases} y_1 = \ell + g + h \\ y_2 = -g - h \\ y_3 = -h \end{cases} \tag{197}$$

y を見付けるには，条件 (c)，

$$\sum x dy - \sum x' dy' = d\varphi \tag{198}$$

$\varphi = 0$ として，

$$Ld\ell + Gdg + Hdh - Ldy_1 - (L - G)dy_2 - (G - H)dy_3 = 0 \tag{199}$$

$$Ld(\ell - y_1 - y_2) + Gd(g + y_2 - y_3) + Hd(h + y_3) = \tag{200}$$

これより，

$$y_1 = \ell + g + h, \quad y_2 = -g - h, \quad y_3 = -h \tag{201}$$

b)

$$\begin{cases} x_1 = H \\ x_2 = L - G \\ x_3 = G - H \end{cases}, \quad \begin{cases} y_1 = \ell + g + h \\ y_2 = \ell \\ y_3 = \ell + g \end{cases} \tag{202}$$

これは，作用変数，角変数である．この y を見付けるのは，

$$Ld\ell + Gdg + Hdh = Hdy_1 + (L - G)dy_2 + (G - H)dy_3 \tag{203}$$

よって，

$$\ell = y_2, \quad g = -y_2 + y_3, \quad h = y_1 - y_3 \tag{204}$$

Kepler 運動では $\dot{g} = \dot{h} = 0 \rightarrow y_1 = y_2 = y_3 = \ell$．このとき，縮退しているという．また，変数の時間微分を取ったとき，

$$\dot{A} \simeq 0 \quad \cdots \quad \text{Slow variable} \tag{205}$$

$$|\dot{A}| > 0 \quad \cdots \quad \text{Fast variable} \tag{206}$$

---

[6]

$$\begin{aligned} L - G &= \sqrt{\mu a} - \sqrt{\mu a(1 - e^2)} \simeq \frac{4}{2}\sqrt{\mu a} e^2 \\ G - H &= \sqrt{\mu a(1 - e^2)}(1 - \cos i) \simeq \sqrt{\mu a(1 - e^2)}\frac{i^2}{2} \end{aligned}$$

という．例えば，
a) では x_1, y_1 は Fast，それ以外は Slow．
b) では x_1, y_1, y_2, y_3 は Fast，それ以外は Slow である．c) Poincaé 変数[7]

$$\begin{cases} x_1 = L \\ x_2 = \sqrt{2(L-G)}\cos(g+h) \\ x_3 = \sqrt{2(G-H)}\cos h \end{cases}, \quad \begin{cases} y_1 = \ell + g + h \\ y_2 = -\sqrt{2(L-G)}\sin(g+h) \\ y_3 = -\sqrt{2(G-H)}\sin h \end{cases} \tag{207}$$

- L：エネルギー関係
- G：全角運動量
- H：角運動量の z 成分

Order-dependence

$$x_2 \sim e\cos\varpi, \quad y_2 \sim -e\sin\varpi$$
$$x_3 \sim i\cos h, \quad y_3 \sim -i\sin h$$

c′)
$(x_2; y_2) \to (-y_2; x_2)$ として，

$$\begin{cases} x_1 = L \\ x_2 = \sqrt{2(L-G)}\sin(g+h) \\ x_3 = \sqrt{2(G-H)}\sin h \end{cases}, \quad \begin{cases} y_1 = \ell + g + h \\ y_2 = \sqrt{2(L+G)}\cos(g+h) \\ y_3 = \sqrt{2(G-H)}\cos h \end{cases} \tag{208}$$

d)
b) に Poincaré 変換を施して，

$$\begin{cases} x_1 = H \\ x_2 = \sqrt{2(L-G)}\cos\ell \\ x_3 = \sqrt{2(G-H)}\cos(\ell+g) \end{cases}, \quad \begin{cases} y_1 = \ell + g + h \\ y_2 = \sqrt{2(L-G)}\sin\ell \\ y_3 = \sqrt{2(G-H)}\sin(\ell+g) \end{cases} \tag{209}$$

1.8.6 自転運動における Delauney 変数

任意の天体について慣性楕円体を考える (2 次のモーメントまで一致する)．慣性主軸 (慣性能率) を $A < B < C$ とする． この ψ, θ, φ を Euler 角という (公転運動における r, θ, φ に対応する)．
共役な momenta を $p_\psi, p_\theta, p_\varphi$ とすると，自転を表す運動方程式は，

$$\frac{d}{dt}(p_\psi, p_\theta, p_\varphi) = \frac{\partial F}{\partial(\psi, \theta, \varphi)}, \quad \frac{d}{dt}(\psi, \theta, \varphi) = -\frac{\partial F}{\partial(p_\psi, p_\theta, p_\varphi)} \tag{210}$$

$$F = -\frac{1}{2}(A\omega_A^2 + B\omega_B^2 + C\omega_C^2) \tag{211}$$

$$\boldsymbol{\omega} = (\omega_A, \omega_B, \omega_c) : (\text{角速度ベクトル}) \tag{212}$$

[7] a) に Poincaré 変換を施す．

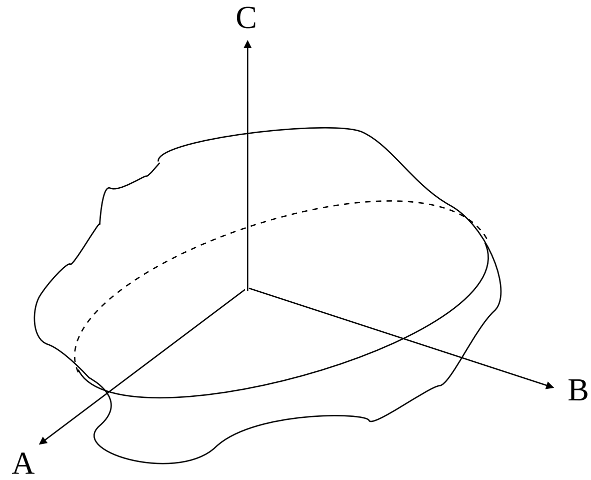

Figure 6: 慣性主軸

自転の角運動量,

$$\boldsymbol{G} = (A\omega_A, B\omega_B, C\omega_C) \tag{213}$$

$$\begin{cases} A\omega_A = p_\theta \cos\varphi + \frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi \cos\theta) \\ B\omega_B = -p_\theta \sin\varphi + \frac{\cos\varphi}{\cos\theta}(p_\psi - p_\varphi \cos\theta) \\ C\omega_C = p_\varphi \end{cases} \tag{214}$$

正準変換,

$$\begin{cases} p_\psi, \psi \\ p_\theta, \theta \\ p_\varphi, \varphi \end{cases} \rightarrow \begin{cases} H = p_\psi \quad (\text{角運動量の } z \text{ 成分}) \\ G = \sqrt{p_\theta^2 + p_\varphi^2 + \frac{1}{\sin^2\theta}(p_\psi + p_\varphi \cos\theta)}^2 \quad (\text{全角運動量}) \\ L = p_\varphi \end{cases} \tag{215}$$

h, g, ℓ を求める.

$$p_\psi = H \tag{216}$$

$$p_\varphi = L \tag{217}$$

$$p_\theta = \pm\sqrt{G^2 - L^2 - \frac{1}{\sin^2\theta}(H - L\cos\theta)^2} \tag{218}$$

母関数,

$$S = S(L, G, H, \psi, \varphi, \theta) \tag{219}$$

$$= L\varphi + H\psi - \int_{\theta^*} \left\{ G^2 - L^2 - \frac{1}{\sin^2\theta}(H - L\cos\theta)^2 \right\}^{\frac{1}{2}} d\theta \tag{220}$$

ここで θ^* は,

$$G^2 - L^2 - \frac{1}{\sin^2\theta}(H - L\cos\theta)^2 = 0 \tag{221}$$

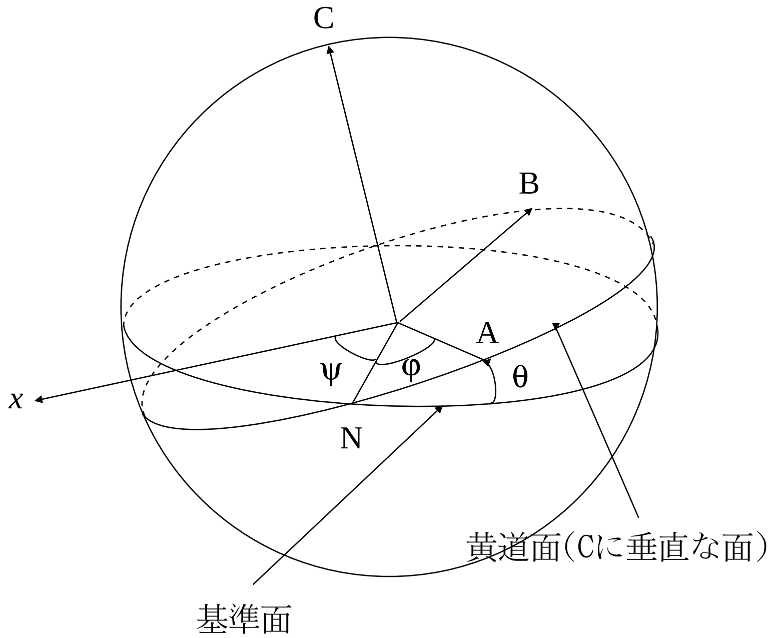

Figure 7: Euler 角

の大きいほうの根．整理すると，

$$G^2\cos^2\theta - 2HL\cos\theta + L^2 - G^2 + H^2 = 0 \tag{222}$$

いま，

$$\frac{H}{G} \equiv \cos I, \quad \frac{L}{G} \equiv \cos J, \quad (0 \le I, J \le \pi) \tag{223}$$

とおくと，

$$\cos^2\theta - 2\cos I\cos J\cos\theta + \cos^2 I + \cos^2 J - 1 = 0 \tag{224}$$

これを解けば，

$$\begin{aligned}\cos\theta &= \cos I\cos J \pm \sqrt{\cos^2 I\cos^2 J - \cos^2 I - \cos^2 + 1} \\ &= \cos I\cos J \pm \sqrt{(\cos^2 I - 1)(\cos^2 J - 1)} \\ &= \cos I\cos J \pm \sin I\sin J \\ &= \cos(I \pm I) \end{aligned} \tag{225}$$

したがって，

$$\theta^* = I + J \tag{226}$$

i)

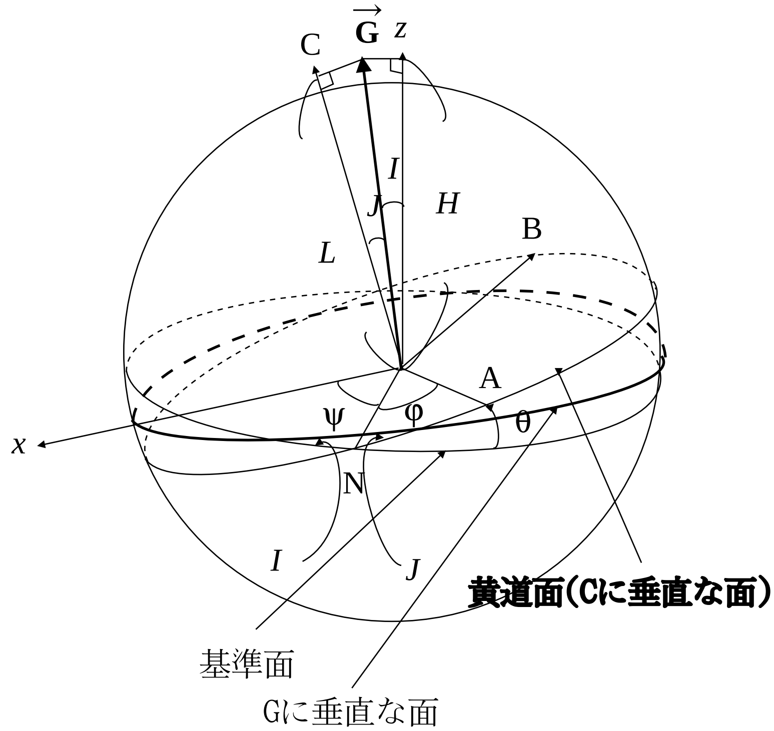

Figure 8: L, H, G, I, J

$$
\begin{aligned}
\ell &= \frac{\partial S}{\partial L} \\
&= \varphi - \int_{\theta^*} \left\{ -L + \frac{\cos\theta}{\sin^2\theta}(H - L\cos\theta) \right\} \\
&\quad \times \left\{ G^2 - L^2 - \frac{1}{\sin^2\theta}(H - L\cos\theta)^2 \right\}^{-\frac{1}{2}} d\theta \\
&= \varphi - \int_{\theta^*} \frac{-\cos J + \frac{\cos\theta}{\sin^2\theta}(\cos I - \cos J\cos\theta)}{\sin J \left\{ 1 - \frac{(\cos I - \cos J \cos\theta)^2}{\sin^2 J \sin^2\theta} \right\}} d\theta \qquad (227)
\end{aligned}
$$

さて，球面三角法より，

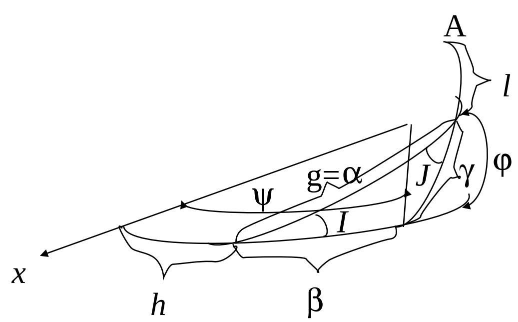

Figure 9: ℓ, g, h

$$\cos I = \cos J \cos\theta + \sin J \sin\theta \cos\gamma$$
$$\cos\gamma = \frac{1}{\sin J \sin\theta}(\cos I - \cos J \cos\theta) \tag{228}$$

これより，積分変数を $\theta \to \gamma$ に変える．

$$\sin\gamma d\gamma = \frac{\cos I \cos\theta - \cos J}{\sin J \sin^{\theta}} d\theta \tag{229}$$

$$\begin{aligned} \text{分母} &= \sin J \sin\gamma \\ \text{分子} &= -\cos J + \sin J \sin\theta d\gamma + \cos J \\ &= \sin J \sin\gamma d\gamma \end{aligned}$$

よって，

$$\ell = \psi - \int_{\gamma^*} d\gamma \tag{230}$$

さて，

$$\cos\gamma^* = \frac{\cos I - \cos J \cos(I+J)}{\sin J \sin(I+J)} = \frac{\cos(J-(I+J)) - \cos J \cos(I+J)}{\sin J \sin(I+J)} = 1 \tag{231}$$
$$\gamma^* = 0 \tag{232}$$

よって，

$$\ell = \varphi - \gamma \tag{233}$$

ii)

$$\begin{aligned} h &= \frac{\partial S}{\partial H} \\ &= \psi + \int_{\theta^*} \frac{H - L\cos\theta}{\sin^2\theta} \left\{ G^2 - L^2 - \left(\frac{H - L\cos\theta}{\sin\theta}\right)^2 \right\}^{-\frac{1}{2}} d\theta \\ &= \psi + \int_{\theta^*} \frac{\cos I - \cos J \cos\theta}{\sin^2\theta} \left\{ \sin^2 J - \frac{(\cos I - \cos J\cos\theta)^2}{\sin^2\theta} \right\}^{-\frac{1}{2}} d\theta \end{aligned} \tag{234}$$

球面三角法より，

$$\cos J = \cos I \cos\theta + \sin J \sin\theta \sin\beta \tag{235}$$
$$\cos\beta = \frac{1}{\sin I \sin\theta}(\cos J - \cos I \cos\theta) \tag{236}$$
$$\sin\beta d\beta = \frac{\cos J \cos\theta - \cos I}{\sin I \sin^2\theta} d\theta \tag{237}$$
$$h = \psi - \int_0 \frac{\sin I \sin\beta}{\sin J \sin\gamma} d\beta \tag{238}$$

正弦定理より,

$$\frac{\sin I}{\sin\gamma} = \frac{\sin J}{\sin\beta} = \frac{\sin\theta}{\sin\alpha} \tag{239}$$

したがって,

$$h = \psi - \beta \tag{240}$$

iii)

$$\begin{aligned} g &= \frac{\partial H}{\partial G} \\ &= -G\int_{\theta^*}\left\{G^2 - L^2 - \frac{(H - L\cos\theta)^2}{\sin^2\theta}\right\}^{-\frac{1}{2}} d\theta \\ &= -\int_{\theta^*}\left\{\sin^2 J - \frac{(\cos I - \cos J\cos\theta)^2}{\sin^2\theta}\right\}^{-\frac{1}{2}} d\theta \\ &= -\int_{\theta^*}\frac{d\theta}{\sin J\sin\gamma} \end{aligned} \tag{241}$$

正弦定理より,

$$= -\int_{\theta^*}\frac{\sin\theta d\theta}{\sin I\sin J\sin\alpha} \tag{242}$$

球面三角法より,

$$-\cos\theta = \cos I\cos J + \sin I\sin J\cos\alpha \tag{243}$$

$$\sin\theta d\theta = -\sin I\sin J\sin\alpha d\alpha \tag{244}$$

したがって,

$$g = \alpha \tag{245}$$

よって,

$$\left\{\begin{array}{l} p_\psi, \psi \\ p_\theta, \theta \\ p_\varphi, \varphi \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{l} H, h \\ G, g \\ L, \ell \end{array}\right. \tag{246}$$

ハミルトニアンは ψ, θ, φ で,

$$F = -\frac{1}{2}\left\{\frac{\left[p_\theta\cos\varphi + \frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta)\right]^2}{A} + \frac{\left[p_\theta\sin\varphi + \frac{\cos\varphi}{\cos\theta}(p_\psi - p_\varphi\cos\theta)\right]^2}{B} + \frac{p_\varphi^2}{C}\right\} \tag{247}$$

新ハミルトニアンは,

$$A\omega_A = G\sin J\sin\ell \tag{248}$$

$$B\omega_B = G\sin J\cos\ell \tag{249}$$

$$C\omega_C = G\cos J = L \tag{250}$$

より，

$$\begin{aligned} F &= -\frac{1}{2}\left\{\frac{G^2}{A}\sin^2 J\sin^2\ell + \frac{G^2}{B}\sin^2 J\cos^2\ell + \frac{L^2}{C}\right\} \\ &= -\frac{1}{2}\left\{\left(\frac{1}{C}-\frac{1}{2A}-\frac{1}{2B}\right)L^2 + \left(\frac{1}{2A}+\frac{1}{2B}\right)G^2 + \left(\frac{1}{2B}-\frac{1}{2A}\right)(G^2-L^2)\cos 2\ell\right\} \end{aligned} \tag{251}$$

ここで G, H はコンスタントであり，ハミルトニアンは非常に簡単になった．
もし $A = B$ なら[8] (回転楕円体で)，

$$F = -\frac{1}{2}\left\{\left(\frac{1}{C}-\frac{1}{A}\right)L^2 + \frac{1}{A}G^2\right\} \tag{252}$$

このとき，

L, G, H, h	Const.
ℓ, g	t に線形 (これが Delauney 型と呼ばれる所以である)

[8] 殆どの天体で $A \sim B$ である．

2　摂動論 (保存系)

2.1　Von Zeipel の方法

2.1.1　人工衛星の運動

運動方程式；

$$\frac{d}{dt}(L,G,H)=\frac{\partial F}{\partial(\ell,g,h)},\quad \frac{d}{dt}(\ell,g,h)=-\frac{\partial F}{\partial(L,G,H)} \tag{253}$$

$$F=\frac{\mu^2}{L^2}+R,\quad R:(\text{摂動関数}) \tag{254}$$

これは,

$$\frac{d^2\rho}{dt^2}=\frac{\mu}{\rho|\rho|}+\frac{\partial R}{\partial\rho} \tag{255}$$

と等価.

もし，地球が球なら $R=0$ (Kepler 運動)．第一近似として回転楕円体 (Spheroid) を考える.

$$R = U-\frac{\mu}{r} \tag{256}$$

$$U : \text{地球のポテンシャル} \tag{257}$$

$$\frac{\mu}{r} : \text{Kepler 運動のときの地球のポテンシャル } U_0 \tag{258}$$

有限体のポテンシャル

$$U = k^2\int_M \frac{\sigma}{\Delta}d\xi d\eta d\zeta \tag{259}$$

$$\Delta = |\vec{\rho}-\vec{r}| \tag{260}$$

$$\vec{\rho} = (\xi,\eta,\zeta) \tag{261}$$

$$\sigma : \text{密度}$$

今 $|\vec{r}|\gg|\vec{\rho}|$

$$\Delta^2=r^2\left[1-2\frac{\vec{r}\cdot\vec{\rho}}{r\rho}\cdot\frac{\rho}{r}+\left(\frac{\rho}{r}\right)^2\right] \tag{262}$$

$\frac{\rho}{r}<1$ で,

$$\frac{1}{\Delta}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^n P_m(\underbrace{\cos\theta}_{\equiv\mu}) \tag{263}$$

$$|P_n(\mu)|\le 1,\quad P_n(1)=1 \tag{264}$$

よって，この級数は絶対収束する[9]．ここで，

$$\begin{aligned}\vec{r} &= (r\cos\beta\cos\varphi, r\cos\beta\sin\varphi, r\sin\beta) \qquad (265)\\ \vec{\rho} &= (\rho\cos\beta_1\cos\varphi_1, \rho\cos\beta_1\sin\varphi_1, \rho\sin\beta_1)\\ \mu &= \cos\theta = \cos\left(\frac{\pi}{2}-\beta\right)\cos\left(\frac{\pi}{2}-\beta_1\right)+\sin\left(\frac{\pi}{2}-\beta\right)\sin\left(\frac{\pi}{2}-\beta_1\right)\cos(\varphi_1-\varphi)\\ &= \sin\beta_1\sin\beta+\cos\beta_1\cos\beta\sin(\varphi_1-\varphi) \qquad (266)\end{aligned}$$

このとき，加法定理より，

$$P_n(\mu) = P_n(\sin\beta)P_n(\sin\beta_1)+2\sum_{n=1}^{\infty}\frac{(n-m)}{(n+m)}P_n^m(\sin\beta)P_n^m(\sin\beta_1)\cos m(\varphi_1-\varphi) \qquad (267)$$

よって，

$$\begin{aligned}U(r,\beta,\varphi) &= k^2\int\frac{dM}{\Delta}\\ &= k^2\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\left[P_n(\sin\beta)\underbrace{\int\rho^n P_n(\sin\beta_1)dM}_{\tilde{C}_n}+2\sum_{m=1}^{n}\frac{1}{r^{n+1}}\frac{(n-m)!}{(n+m)!}P_n^m(\sin\beta)\right.\\ &\qquad\left.\times\left\{\cos m\varphi\underbrace{\int\rho^n P_n^m(\sin\beta_1)\cos m\varphi_1 dM}_{\tilde{C}_{nm}}+\sin m\varphi\underbrace{\int\rho^n P_n^m(\sin\beta_1)\sin m\varphi_1 dM}_{\tilde{S}_{nm}}\right\}\right]\end{aligned} \qquad (268)$$

結局，

$$\begin{aligned}U(r,\beta,\varphi) &= \sum_{n=0}^{\infty}\frac{k^2}{r^{n+1}}\left[\tilde{C}_nP_n(\sin\beta)+\tilde{C}_{nm}P_n^m(\sin\beta)\cos m\varphi+\tilde{S}_{nm}P_n^m\sin(\beta)\sin m\varphi\right]\\ &= \frac{k^2M}{r}+\frac{k^2M}{r}\sum_{n=0}^{\infty}\left[C_n\left(\frac{R}{r}\right)^nP_n(\sin\beta)+C_{nm}\left(\frac{R}{r}\right)^nP_n^m(\sin\beta)\cos m\varphi\right.\\ &\qquad\left.+S_{nm}\left(\frac{R}{r}\right)^nP_n^m(\sin\beta)\sin m\varphi\right] \qquad (269)\\ C_n &= \frac{1}{MR^n}\int\rho^nP_n(\sin\beta_1)dM \qquad (270)\\ \begin{pmatrix}C_{nm}\\ S_{nm}\end{pmatrix} &= \frac{1}{MR^n}\frac{2(n-m)}{(n+m)}\int\rho^nP_n^m(\sin\beta_1)\begin{pmatrix}\cos m\varphi_1\\ \sin m\varphi_1\end{pmatrix}dM \qquad (271)\end{aligned}$$

R : 天体の代表的な長さ

9

$$P_0(\mu)=1,\quad P_1(\mu)=\mu,\quad P_2(\mu)=\frac{1}{2}(3\mu^2-1),\quad P_3(\mu)=\frac{1}{2}(5\mu^2-3\mu)$$

地球の場合，

$$C_2 \sim 10^{-3}, \quad C_n = C_{nm} = S_{nm} \sim 10^{-6}$$

主要項

$$U = \underbrace{U_0}_{\text{Kepler 項}} + U_1 + U_2 + U_3 + \cdots \tag{272}$$

$$U_i : i \text{ 次の調和項} \tag{273}$$

1. 1次

$$C_1 = \frac{1}{MR}\int \rho \sin\beta_1 dM = \frac{1}{MR}\int \zeta dM = 0 \tag{274}$$

$$C_{11} = \frac{1}{MR}\int \cos\beta_1 \cos\varphi_1 dM = \frac{1}{MR}\int \xi dM = 0 \tag{275}$$

$$S_{11} = \frac{1}{MR}\int \rho\cos\beta_1 \sin\varphi_1 dM = \frac{1}{MR} int \eta dM = 0 \tag{276}$$

$$U_1 = 0 \quad (\text{重心が原点にあるから}) \tag{277}$$

2. 2次

$$P_2^1(\sin\beta_1) = 3\sin\beta_1\cos\beta_1, \quad P_2^2(\sin\beta_1) = 3\cos^2\beta_1 \tag{278}$$

とすると，

$$\rho^2 P_2(\sin\beta_1) = \zeta^2 - \frac{1}{2}(\xi^2+\eta^2)$$

$$\rho^2 P_2^1(\sin\beta_1)\begin{pmatrix}\cos\varphi_1\\ \sin\varphi_1\end{pmatrix} = 3\zeta \times \begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{279}$$

$$\rho^2 P_2^2(\sin\beta_1)\begin{pmatrix}\cos\varphi_1\\ \sin\varphi_1\end{pmatrix} = 3 \times \begin{pmatrix}\xi^2-\eta^2\\ 2\xi\eta\end{pmatrix} \tag{280}$$

慣性モーメント

$$A = \int(\eta^2+\zeta^2)dM, \quad B = \int(\zeta^2+\xi^2)dM, \quad C = \int(\xi^2+\eta^2)dM \tag{281}$$

慣性積

$$D = \int \eta\zeta dM, \quad E = \int \zeta\xi dM, \quad F = \int \xi\eta dM$$

とすると，

$$C_2 = \frac{1}{MR^2}\frac{1}{2}(A+B-2C) \tag{282}$$

$$\begin{pmatrix}C_{21}\\ S_{21}\end{pmatrix} = \frac{1}{MR^2}\begin{pmatrix}E\\ D\end{pmatrix} \tag{283}$$

$$\begin{pmatrix}C_{22}\\ S_{22}\end{pmatrix} = \frac{1}{MR^2}\begin{pmatrix}B-A\\ 2F\end{pmatrix} \tag{284}$$

よって，

$$U_2 = \frac{k^2MR^2}{r^3}\Big[C_2P_2(\sin\beta) + C_{21}P_2^1(\sin\beta)\cos\varphi + S_{21}P_2^1(\sin\beta)\sin\varphi + C_{22}P_2^2(\sin\beta)\cos 2\varphi + S_{22}P_2^2(\sin\beta)\sin 2\varphi\Big] \quad (285)$$

慣性主軸を 3 軸にとると $D = E = F = 0$ とすることが出来る．このとき A, B, C を主慣性モーメントという．すると，

$$U_2 = \frac{k^2MR^2}{r^3}\left[\left(C - \frac{A+B}{2}\right)\left(\frac{1}{2} - \frac{3}{2}\sin^2\beta\right) - \frac{3}{4}(A-B)\cos^2\beta\cos 2\varphi\right] \quad (286)$$

3. 3 次

$$P_3^1(\sin\beta_1) = \frac{3}{2}\cos\beta_1(5\sin\beta_1 - 1) \quad (287)$$

$$P_3^2(\sin\beta_1) = 15\cos^2\beta_1\sin\beta_1 \quad (288)$$

$$P_3^3(\sin\beta_1) = 15\cos^3\beta_1 \quad (289)$$

すると，

$$\rho^3P_3(\sin\beta_1) = \frac{5}{2}\zeta^2 - \frac{3}{2}\zeta(\zeta^2+\eta^2+\xi^2) = \zeta^3 - \frac{3}{2}\zeta(\xi^2+\eta^2) \quad (290)$$

$$\rho^3P_3^1(\sin\beta_1)\begin{pmatrix}\cos\varphi_1\\ \sin\varphi_1\end{pmatrix} = \frac{3}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\left[4\zeta^2 - (\xi^2+\eta^2)\right] \quad (291)$$

$$\rho^3P_3^2(\sin\beta_1)\begin{pmatrix}\cos 2\varphi_1\\ \sin 2\varphi_1\end{pmatrix} = 15\zeta\times\begin{pmatrix}\xi^2-\eta^2\\ 2\xi\eta\end{pmatrix} \quad (292)$$

$$\rho^3P_3^3(\sin\beta_1)\begin{pmatrix}\cos 3\varphi_1\\ \sin 3\varphi_1\end{pmatrix} = 15\times\begin{pmatrix}\xi^3-3\xi\eta^2\\ -\eta^3+3\eta\xi^2\end{pmatrix} \quad (293)$$

結局 z 軸に対して対称なら，

$$U_3 = \frac{k^2MR^3}{r^4}\left[C_3\left(\frac{5}{2}\sin^3\beta - \frac{3}{2}\sin\beta\right) + C_{32}\cdot 15\cos^2\beta\sin\beta\cos 2\varphi\right] \quad (294)$$

摂動の続き

$$R = U - U_0 = U_2 + U_3 \quad (295)$$

とおく．

$$U = \frac{\mu}{r}\left[1 - J_2\frac{R^2}{r^2}P_2(\sin\beta) + J_3\frac{R^3}{r^3}P_2(\sin\beta)\right] \quad (296)$$

地球は，oblete spheroid 偏球 (Pan-Cake 型)[10]．

$$F = \frac{\mu^2}{2L^2} - \frac{\mu J_2R^2}{r^3}P_2(\sin\beta) + \frac{\mu J_3R^3}{r^4}P_3(\sin\beta) \quad (297)$$

$$= F_0 + F_2 + F_3 \quad (298)$$

10

$$J_2 \sim 10^{-3}, \quad J_3 \sim 10^{-6}$$

つまり，

$$J_3 = J_2^2 \tag{299}$$

と考える．

$$\begin{aligned} P_2(\sin\beta) &= \frac{3}{2}\sin^2\beta - \frac{1}{2} && (300) \\ \sin\beta &= \sin i \sin(f + \underbrace{\omega}_{g}), \quad \cos i = \frac{H}{G} && (301) \\ \frac{3}{2}\sin^2\beta &= \frac{3}{4}(1-\cos^2 i)(1-\cos(2f+2g)) && (302) \\ P_2(\sin\beta) &= \frac{3}{4}(1-\cos^2 i)(1-\cos(2f+2g)) - \frac{1}{2} && (303) \end{aligned}$$

よって，

$$F_1 = \frac{\mu J_2 R^2}{2a^3}\left[\left(-\frac{1}{2}+\frac{3}{2}\cos^2 i\right)\frac{a^3}{r^3} + \left(\frac{3}{2}-\frac{3}{2}\cos^2 i\right)\frac{a^3}{r^3}\cos(2f+2g)\right] \tag{304}$$

同様にして，

$$\begin{aligned} F_2 &= \frac{\mu J_3 R^3}{2a^4}\left[\left(\frac{3}{4}-\frac{15}{4}\cos^2 i\right)\sin i \frac{a^4}{r^4}\sin(f+g)\right. \\ &\quad \left. -\left(\frac{5}{4}-\frac{5}{4}\cos^2 i\right)\sin i \frac{a^4}{r^4}\sin(3f+3g)\right] && (305) \end{aligned}$$

さて，e^4 までとると，

$$\begin{aligned} \frac{a^3}{r^3} &= 1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \left(3e + \frac{27}{8}e^3\right)\cos\ell \\ &+ \left(\frac{9}{2}e^2 + \frac{7}{2}e^4\right)\cos 2\ell + \frac{53}{8}e^3\cos 3\ell + \frac{77}{8}e^4\cos 4\ell + \mathcal{O}(e^5) && (306) \\ &\left(\frac{G}{L} = \sqrt{1-e^2}\right) \\ \frac{a^3}{r^3}\cos(2f+2g) &= \left(-\frac{e}{2}+\frac{e^3}{16}\right)\cos(\ell+2g) + \frac{e^3}{48}\cos(-\ell+2g) \\ &+ \left(1 - \frac{5}{2}e^2 + \frac{13}{16}e^4\right)\cos(2\ell+2g) + \frac{e^4}{24}\cos(-2\ell+2g) \\ &+ \left(\frac{7}{2}e - \frac{123}{16}e^3\right)\cos(3\ell+2g) + \left(\frac{17}{2}e^2 - \frac{115}{6}e^4\right)\cos(4\ell+2g) \\ &\frac{845}{48}e^3\cos(5\ell+2g) + \frac{533}{16}e^4\cos(6\ell+2g) + \mathcal{O}(e^5) && (307) \\ \frac{a^4}{r^4}\sin(\ell+g) &= \left(e + \frac{5}{2}e^3\right)\cos g + (\ell\text{の周期項}) && (308) \\ \frac{a^4}{r^4}\sin(3\ell+3g) &= (\ell\text{の周期項}) && (309) \end{aligned}$$

※ D'Alembert's charactaristics : $\cos(n\ell + mg)$ の係数は $e^{|n-m|}$ から始まる．
明らかに F_1, F_2 は h を含まないので $H = \text{Const.}$ (積分)．h は ℓ, g, L, G が決まれば決定される．これは軸対称の仮定から当たり前．

2.1.2 Von Zeipel の方法

正準変換で解く．

$$L, G, H, \ell, g, h \rightarrow L', G', H', \ell', g', h' \tag{310}$$

新ハミルトニアンを ℓ', h' を含まないようにすることが出来る．これを ℓ', h' を消去するという．

母関数，

$$S = S(L', G', H'; \ell, g, -), \quad (h \text{ は } F \text{ に含まれない}) \tag{311}$$

$$S = \underbrace{L'\ell + G'g + H'h}_{\text{恒等変換}} + S_1 + S_2 \tag{312}$$

とおける．ここで，

$$S_1 \simeq O(J_2), \quad S_2 \simeq O(J_3) \tag{313}$$

変換

$$\begin{aligned} (L, G, H) &= \tfrac{\partial S}{\partial(\ell, g, h)} = (L', G', H') + \tfrac{\partial S_1}{\partial(\ell, g, h)} + \tfrac{\partial S_2}{\partial(\ell, g, h)} \\ (\ell', g', h') &= \tfrac{\partial S}{\partial(L', G', H')} = (\ell, g, h) + \tfrac{\partial S_1}{\partial(L', G', H')} + \tfrac{\partial S_2}{\partial(L', G', H')} \end{aligned} \tag{314}$$

変換の目的：ℓ の消去

$$F(L, G, H, \ell, g, -) = F(L', G', H', -, g', -) \tag{315}$$

$$F = F_0(L) + F_1(L, G, H, \ell, g, -) + F_2(L, G, H, \ell, g, -) + F_2(L, G, H, \ell, g, -) + F_2(L, G, H, \ell, g, -) \tag{316}$$

$$F^* = F_0(L') + F_1^*(L', G', H', -, g', -) \tag{317}$$

$$F = F^* \tag{318}$$

この式に (314) を代入して次数を比較する．
3 次以上を無視すれば，

$$\begin{aligned} & F_0\left(L' + \frac{\partial S_1}{\partial \ell} + \frac{\partial S_2}{\partial \ell}\right) + F_1\left(L' + \frac{\partial S_1}{\partial \ell}, G' + \frac{\partial S_2}{\partial g}, H', \ell, g, -\right) + F_2\left(L', G', H', \ell, g, -\right) \\ = \; & F_0^*(L') + F_1^*(L', G', H', -, g', -) + F_2^*(L', G', H', -, g', -) \end{aligned} \tag{319}$$

$$(0\text{ 次}) \qquad F_0(L) = F_0^*(L') \tag{320}$$

$$(1\text{ 次}) \qquad \frac{dF_0}{dL'}\frac{\partial S_1}{\partial \ell} + F_1(L', G', H', \ell, g, -) = F_1^*(L', G', H', -, g', -) \tag{321}$$

$$\begin{aligned} (2\text{ 次}) \qquad & \frac{dF_0}{dL'}\frac{\partial S_2}{\partial \ell} + \frac{1}{2}\frac{d^2F_0}{dL'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{dF_1}{dL'}\frac{\partial S_1}{\partial \ell} + \frac{dF_1}{dG'}\frac{\partial S_1}{\partial g} \\ & + F_2(L', G', H', \ell, g, -) = F_2^*(L', G', H', -, g', -) \end{aligned} \tag{322}$$

これより，

$$F_0^* = \frac{\mu^2}{2L'^2} \tag{323}$$

n 次で S_n, F_n^* を決定しなければならない →しかし，F_n^* には ℓ を含まないようにとるため，S_n, F_n^* は一意に決定される．

$$(1\,次)\quad F_1^*(L', G', H', -, g, -) = \langle F_1(L', G', H', \ell, g, -)\rangle_\ell \tag{324}$$

ここで $\langle ...\rangle_\ell$ は ℓ に対する平均．ここで，この後 F_1^* の $g \to g'$ としなければならい．このような方法を平均の原理という[11]．
ただちに，

$$S_2 = -\left(\frac{dF_0}{dL'}\right)^{-1} \int (F_1 - \langle F_1\rangle_\ell)\, d\ell \tag{326}$$

$$\begin{aligned}(2\,次)\quad F_2^* &= \tilde{F}_2 \\ &= \left\langle \frac{1}{2}\frac{d^2F_0}{dL'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} - \frac{\partial F_1}{\partial g}\frac{\partial S_1}{\partial G'} + F_2 \right\rangle_{\ell, g\to g'}\end{aligned} \tag{327}$$

$$S_2 = -\left(\frac{dF_1}{dL'}\right)^{-1} \int \left(\tilde{F}_2 - \langle \tilde{F}_2\rangle_\ell\right) d\ell \tag{328}$$

ただし，

$$F_0(L') = \frac{\mu^2}{2L'^2} \to \begin{cases} \frac{dF_0}{dL'} = -\frac{\mu^2}{L'^3} \equiv -n \\ \frac{d^2F_0}{dL'^2} = \frac{3\mu^2}{L'^4} = \frac{3}{a^2} \end{cases} \tag{329}$$

ここで，a, n はもちろん L' の関数である．

2.1.3　1次の摂動

ここに出てくる e, a 等は全て $L', G', H', ...$ の関数である．

$$F_1^* = \langle F_1\rangle_{g\to g'} = \frac{\mu J_2 R^2}{2a^3} A\left(1 + \frac{3}{2}e^2 + \frac{15}{8}e^4\right) \tag{330}$$

$$A \equiv -\frac{1}{2} + \frac{3}{2}\cos^2 i \tag{331}$$

$$B \equiv \frac{3}{2} - \frac{3}{2}\cos^2 i \tag{332}$$

$$n\frac{\partial S_1}{\partial \ell} = F_1 - \langle F_1\rangle \tag{333}$$

[11] 平均値法
目的は $\langle F\rangle_t$ の評価．しかし，

$$\rangle F\langle_t = \rangle F_0\langle_\ell + \rangle F_1\langle_\ell + \rangle F_2\langle_\ell + \cdots \tag{325}$$

というように，次々の平均エネルギーを求め続ける．

積分すると，

$$
\begin{aligned}
S_1 &= \frac{\mu J_2 R^2}{2a^3 n}\left[A\left\{\left(3e+\frac{27}{8}e^3\right)\cos\ell+\left(\frac{9}{2}e^2+\frac{7}{2}e^4\right)\cos 2\ell+\frac{53}{8}e^3\cos 3\ell+\frac{77}{8}e^4\cos 4\ell\right\}\right. \\
&+B\left\{\left(-\frac{e}{2}+\frac{e^3}{16}\right)\cos(\ell+2g)+\frac{e^3}{48}\cos(-\ell+2g)+\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4\right)\cos(2g+2\ell)\right. \\
&+\frac{e^4}{24}\cos(-2\ell+2g)+\left(\frac{7}{2}e-\frac{123}{16}\right)\cos(3\ell+2g)+\left(\frac{17}{2}e^2-\frac{115}{6}e^4\right)\cos(4\ell+2g) \\
&\left.\left.+\ \frac{845}{48}e^3\cos(5\ell+2g)+\frac{533}{16}e^4\cos(6\ell+2g)\right\}\right] \qquad (334)
\end{aligned}
$$

$\frac{\partial}{\partial G'}$ などの項は，

$$
\frac{\partial}{\partial G'}=\frac{\partial e}{\partial G'}\frac{\partial}{\partial e}=-\frac{G'}{eL'^2}\frac{\partial}{\partial e} \qquad (335)
$$

となって，$\frac{\partial}{\partial G'}$ で e が 2 次落ちる．

一般に，S_n で $e^{2(n-1)}$ 次落ちる → e を高次まで取らなければならない[12]．計算すると[13]，

$$
\begin{aligned}
F_2^* &= \frac{\mu^2 J_2^2 R^4}{4n^2 a^8}\left[\frac{3}{8}(1-8\cos^2 i+19\cos^4 i)+\frac{3}{32}(7-8\cos^2 i+243\cos^4 i)e^2\right. \\
&\left.+\ \frac{3}{4}\left(1-\frac{17}{2}\cos^2 i+\frac{15}{2}\cos^4 i\right)e^2\cos 2g'\right] \\
&+\frac{3\mu J_2 R^3}{8a^4}(1-5\cos^2 i)\sin i\left(e+\frac{5}{2}e^3\right)\sin g' \qquad (336)
\end{aligned}
$$

新運動方程式（長周期摂動）

$$
\frac{d}{dt}(\ell', g', h') = -\frac{\partial F^*}{\partial(L', G', H')} \qquad (337)
$$

$$
\frac{d}{dt}(\text{Ł}', G', H') = \frac{\partial F^*}{\partial(\ell', g', h')} \qquad (338)
$$

$$
F^* = \frac{\mu^2}{2L'^2}+\frac{\mu J_2 R^2}{2a^3}A(1-e^2)^{-3/2}+F_2^*(L', G', H', -, g', -) \qquad (339)
$$

---

[12] もし，e で展開しないとどうなるか？

$$
\begin{aligned}
\left\langle\frac{a^3}{r^3}\right\rangle &= \frac{1}{\pi}\int_0^\pi \frac{a^3}{r^3}d\ell \quad \left(d\ell=\frac{1}{\sqrt{1-e^2}}\frac{r^2}{a^2}df\right) \\
&= \frac{1}{\pi}\frac{1}{\sqrt{1-e^2}}\int_0^\pi \frac{a}{r}df=\frac{1}{\pi}\frac{1}{(1-e^2)^{3/2}}\int_0^\pi(1+e\cos f)df=(1-e^2)^{-3/2}
\end{aligned}
$$

例えば，

$$
\int\left(\frac{a^3}{r^3}-\left\langle\frac{a^3}{r^3}\right\rangle\right)d\ell=\int\frac{a^3}{r^3}d\ell-\left\langle\frac{a^3}{r^3}\right\rangle\ell=(1-e^2)^{-3/2}\left[\int(1+e\cos f)df-\ell\right]=(1-e^2)^{-3/2}(f+e\sin f-\ell)
$$

すると，

$$
S_1=\frac{\mu J_2 R^2}{2na^3}\left[A(1-e^2)^{-3/2}(f+e\sin f-\ell)+B\cdot\ldots\right]
$$

と正確に求まる．

[13] 実は F_n^* から $n-1$ 次の長周期摂動が出るので，n 次の摂動では，S_n, F_{n+1}^* を求める．

積分

$$L' = \text{const.}, \quad H' = H = \text{const.} \tag{340}$$

明らかに，

$$\frac{dg'}{dt} = \mathcal{O}(J_2) \tag{341}$$

→ ゆっくりしているので長周期摂動という．

これをもう一度正準変換して g' を消去する．そのとき，F_0^* は const. なので，F_1^* が F_0 の役割をする．よって，F_2^* が実は長周期摂動での主要項となる．

2.1.4 g' の消去

正準変換

$$(L', G', H', \ell', g', h') \to (L'', G'', H'', \ell'', g'', h'') \tag{342}$$

母関数

$$S^* = L''\ell' + G''g' + H''h' + S_1^*(L'', G'', H'', -, g', -) + S_2^* \tag{343}$$

変換

$$(L', G', H') = \frac{\partial S^*}{\partial(\ell', g', h')} = (L'', G'', H'') + \frac{\partial S_1^*}{\partial(\ell', g', h')} + \frac{\partial S_2^*}{\partial(\ell', g', h')} \tag{344}$$

$$(\ell'', g'', h'') = \frac{\partial S^*}{\partial(L'', G'', H'')} = (\ell', g', h') + \frac{\partial S_1^*}{\partial(L'', G'', H'')} + \frac{\partial S_2^*}{\partial(L'', G'', H'')} \tag{345}$$

目的

$$F^*(L', G', H', -, g', -) = F^{**}(L'', G'', H'', -, -, -) \tag{346}$$

$$F^* = F_0^*(L') + F_1^*(L', G', H', -, g', -) + F_2^*(L', G', H', -, g', -) \tag{347}$$

$$F^{**} = F_0^{**}(L'') + F_1^{**}(L'', G'', H'') + F_2^{**}(L'', G'', H'') \tag{348}$$

明らかに $L' = L'', H' = H''$ より直ちに $F_0^*(L') = F_0^{**}(L')$ (0 次)．代入すれば，

$$F^*\left(L'', G'' + \frac{\partial S_1^*}{\partial g'}, H''\right) + F_2^*(L'', G'', H'', -, g', -) \tag{349}$$

$$= \quad F_1^{**}(L'', G'', H'') + F_2^{**}(L'', G'', H'') \tag{350}$$

$$(1\,\text{次}) \qquad F_1^*(L'', G'', H') = F_1^{**}(L'', G'', H') \tag{351}$$

$$(2\,\text{次}) \qquad \frac{\partial F_1^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_2^*(L'', G'', H'', -, g', -) = F_2^{**}(L'', G'', H'') \tag{352}$$

これより，

$$F_0^{**} = \frac{\mu^2}{2L''^2} \tag{353}$$

$$F_1^{**} = \frac{\mu J_2 R^2}{2s^3}(1-e)^{-3/2}\left(-\frac{1}{2}+\frac{3}{2}\cos^2 I\right) \tag{354}$$

勿論，a, e, I 等は $''$ の変数で表されている．
平均値法より，

$$\begin{aligned} F_2^{**} &= \langle F_2^*(L'', G'', H'', -, g', -)\rangle_{g'} \\ &= \frac{\mu^2 J_2^2 R^4}{4n^2a^8}\left[\frac{3}{8}(1-8\cos^2 i+19\cos^4 i)+\frac{3}{32}(7-82\cos^2 i+243\cos^4 i)e^2\right] \end{aligned} \tag{355}$$

よって，

$$S_1^* = -\left(\frac{\partial F_1^*}{\partial G''}\right)^{-1}\int (F_2^* - \langle F_2^*\rangle_{g'})dg' \tag{356}$$

さて，

$$F_1^* = \frac{\mu J_2 R^2}{2a^3}\frac{L''^3}{G''^3}\left(-\frac{1}{2}+\frac{3}{2}\frac{H''^2}{G''^2}\right) \tag{357}$$

$$\begin{aligned} \frac{\partial F_1^*}{\partial G''} &= \frac{\mu J_2 R^2}{2a^3}L''^3\left(\frac{3}{2}\frac{1}{G''^4}-\frac{15}{2}\frac{H''^2}{G''^5}\right) \\ &= \frac{\mu J_2 R^2}{2a^3 L''}(1-e^2)^{-2}\frac{3}{2}(1-5\cos^2 i) \quad (L''=na^2) \\ &= \frac{3\mu J_2 R^2}{2na^5}(1-e^2)^{-2}(1-5\cos^2 i) \end{aligned} \tag{358}$$

ここで，

$$\cos i = \frac{1}{\sqrt{5}} \longrightarrow i \sim 63^\circ \quad \text{(Critical Inclination)} \tag{359}$$

このときは，

$$\frac{\partial F_1^*}{\partial G''} \leq \mathcal{O}(J_2^{3/2}) \tag{360}$$

となりうるので，別の方法を用いねばならない．

$$S_1^* = -\frac{\mu J_2 R^2}{8na^3}\frac{1-\frac{17}{2}\cos^2 i+\frac{15}{2}\cos^4 i}{1-5\cos^2 i}e^2\sin 2g' + \frac{J_3 Rna}{2J_2}\sin i\left(e+\frac{e^3}{2}\right)\cos g' \tag{361}$$

新運動方程式

$$\frac{d}{dt}(L'', G'', H'') = \frac{\partial F^{**}}{\partial(\ell'', g'', h'')} \tag{362}$$

$$\frac{d}{dt}(\ell'', g'', h'') = -\frac{\partial F^{**}}{\partial(L'', G'', H'')} \tag{363}$$

$$F^{**} = \frac{\mu^2}{2L''^2}+\frac{\mu J_2 R^2}{2a^3}(1-e^2)^{-3/2}\left(-\frac{1}{2}+\frac{3}{2}\cos^2 i\right)+F_2^{**} \tag{364}$$

解

$$(L'', G'', H'') = (L'', G'', H'')_0 \quad (365)$$

$$(\ell'', g'', h'') = -\frac{\partial F^{**}}{\partial (L'', G'', H'')} t + \underbrace{(\ell'', g'', h'')_0}_{\text{const.}} \quad (366)$$

2.1.5 長周期摂動と短周期摂動と長年摂動

$$\ell'' = -\left(\frac{\partial F^{**}}{\partial L''}\right) t + \ell''_0 \quad (367)$$

$$-\frac{\partial F^{**}}{\partial L''} = n + \frac{3nJ_2R^2}{2a^2}\left(-\frac{1}{2}+\frac{3}{2}\cos^2 i\right)(1-e^2)^{-3/2} + \frac{3nJ_2^2R^4}{4a^4}\left(\frac{13}{16}-\frac{39}{8}\cos^2 i+\frac{137}{16}\cos^4 i\right) \quad (368)$$

$$g'' = -\frac{\partial F^{**}}{\partial G''} + g''_0 \quad (369)$$

$$-\frac{\partial F^{**}}{\partial G''} = \frac{3nJ_2R^2}{2a^2}\left(-\frac{1}{2}+\frac{5}{2}\cos^2 i\right)(1-e^2)^{-2} + \frac{3nJ_2^2R^4}{4a^4}\left(\frac{7}{16}-\frac{57}{8}\cos^2 i+\frac{395}{16}\cos^4 i\right) \quad (370)$$

$$k'' = -\frac{\partial F^{**}}{\partial H''} t + h''_0 \quad (371)$$

$$-\frac{\partial F^{**}}{\partial H''} = \frac{3nJ_2R^2}{2a^2}\cos i(1-e^2)^{-2} + \frac{3nJ_2^2R^4}{4a^4}\cos i\left(2-\frac{19}{2}\cos^2 i\right) \quad (372)$$

これは長年摂動である[14]．これら全ては近地点からの要素で，$''$ の関数である．F_2 は 4 次までとったのに，これでは e^0 までしか取れない！
例えば，赤道付近の人工衛星で $\cos i = 0$

$$g'' \nearrow, \quad h'' \searrow. \quad g'' + h'' \nearrow \quad (373)$$

$$\Delta \ell'' + g'' + h'' \sim \frac{3nJ_2R^2}{a^2} \quad (374)$$

よって円運動は加速される（ふくらみのため）．
長周期摂動（1 次）

$$L' = L'' \quad (375)$$

$$G' = G'' + \frac{\partial S_1^*}{\partial g''} = G'' - \frac{nJ_2R^2}{4}\frac{1-\frac{17}{2}\cos^2 i+\frac{15}{2}\cos^4 i}{1-5\cos i}\cos 2g'' - \frac{J_3R}{2J_2}na\sin i\left(e+\frac{e^3}{2}\right)\sin g'' \quad (376)$$

$$H' = H'' \quad (377)$$

$$\ell' = \ell'' - \frac{\partial S_1^*}{\partial L''} = \ell'' + \frac{J_2R^2}{4a^2}\frac{1-\frac{17}{2}\cos^2 i+\frac{15}{2}\cos^4 i}{1-5\cos^2 i}\sin 2g'' - \frac{J_3R^2}{2J_2a}\sin i\left(\frac{1}{e}-\frac{e}{2}\right)\cos g'' \quad (378)$$

[14] $e \to 0, i \to 0$ で それぞれ $(\ell, g), (g, h)$ は定義されないのに，どうして確定しているのか？

$$
\begin{aligned}
g' &= g'' - \frac{\partial S_1^*}{\partial G''} \\
&= g'' - \frac{J_2 R^2}{4a^2} \frac{1 - \frac{17}{2}\cos^2 i + \frac{15}{2}\cos^4 i}{1 - 5\cos^2 i} \sin 2g' + \frac{J_3 R}{2J_2 a}\left[\sin i\left(\frac{1}{e} + e\right) - \frac{\cos^2 i}{\sin i} e\right]\cos g''
\end{aligned} \tag{379}
$$

$$
\begin{aligned}
h'' &= h'' - \frac{S_1^*}{\partial H''} \\
&= h'' + \mathcal{O}(J_2 e^2) + \frac{J_3 R}{2J_2 a}\frac{\cos i}{\sin i} e \cos g''
\end{aligned} \tag{380}
$$

注意

$$
\frac{1}{e}, \quad \frac{1}{\sin i}
$$

があるから，

- $e \sim 0$ では ℓ, g が決まりにくい．例えば，

$$
\ell' + g' = \ell'' + g'' + \frac{J_3 R^2}{2J_2 a}\left[\sin i \frac{3}{2} e - \frac{\cos^2 i}{\sin i} e\right]\cos g'' \tag{381}
$$

- $i \sim 0$ では g, h が決まりにくい．例えば，

$$
g' + h' = g'' + h'' - \frac{J_2 R^2}{4a^2}\frac{1 - \frac{17}{2}\cos^2 i + \frac{15}{2}\cos^4 i}{1 - 5\cos^2 i}\sin 2g'' + \frac{J_3 R}{2J_2 a}\sin i\left(\frac{1}{e} + e\right)\cos g'' \tag{382}
$$

  さらに，

$$
\ell' + g' + h' = \ell'' + g'' + h'' + \frac{J_3 R}{2J_2 a}\frac{3}{2} e \sin i \cos g'' \tag{383}
$$

  とうまくいっている．これは g'' ($\dot{g}'' \sim \mathcal{O}(J_2)$) → 長周期摂動

短周期摂動（1 次）[15]

$$
\begin{aligned}
L &= L' + \frac{\partial S_1}{\partial \ell'} \\
&= L'' + \frac{nJ_2 R^2}{2}\left[\left(-\frac{1}{2} + \frac{3}{2}\cos^2 i\right) 3e\cos\ell'' + \left(\frac{3}{2} - \frac{3}{2}\cos^2 i\right)\right. \\
&\quad \left.\times\left\{-\frac{e}{2}\cos(\ell'' + 2g'') + \cos(2\ell'' + 2g'') + \frac{7}{2}e\cos(3\ell'' + 2g'')\right\}\right]
\end{aligned} \tag{384}
$$

$$
\begin{aligned}
G &= G' + \frac{\partial S_1}{\partial g'} \\
&= G' + \frac{nJ_2 R^2}{2}\left(\frac{3}{2} - \frac{3}{2}\cos^2 i\right)\left[-e\cos(\ell'' + 2g'') + \cos(2\ell'' + 2g'') + \frac{7}{2}e\cos(3\ell'' + 2g'')\right]
\end{aligned} \tag{385}
$$

$$
H = H' \tag{386}
$$

$$
\ell = \ell' - \frac{J_2 R^2}{2a^2}\left[\left(-\frac{1}{2} + \frac{3}{2}\cos^2 i\right)\left\{\left(\frac{3}{e} - \frac{15}{8}e\right)\sin\ell'' + \frac{9}{2}\sin 2\ell'' + \frac{53}{8}e\sin 3\ell''\right\}\right.
$$

[15] 本当なら要素はすべて ℓ', g' の関数だが，1 次まで取るなら ℓ'', g'' の関数としてよい．

$$+\left(\frac{3}{2}-\frac{3}{2}\cos^2 i\right)\left\{\left(-\frac{1}{2e}+\frac{35}{16}e\right)\sin(\ell''+2g'')-\frac{e}{16}\sin(-\ell''+2g'')-4\sin(2\ell''+2g'')\right.$$
$$\left.\left.+\left(\frac{7}{6e}-\frac{593e}{48}\right)\sin(3\ell''+2g'')+\frac{17}{4}\sin(4\ell''+2g'')+\frac{169}{16}e\sin(5\ell''+2g'')\right\}\right] \tag{387}$$

$$\begin{aligned} g &= g'+\frac{J_2R^2}{2a^2}\left[\left(-\frac{1}{2}+\frac{3}{2}\cos^2 i\right)\left\{\left(\frac{3}{e}+\frac{69}{8}e\right)\sin\ell''+\frac{9}{2}\sin 2\ell''+\frac{53}{8}e\sin 3\ell''\right\}\right. \\ &\quad +\left(\frac{3}{2}-\frac{3}{2}\cos^2 i\right)\left\{\left(-\frac{1}{2e}+\frac{7}{16}e\right)\sin(\ell''+2g'')-\frac{e}{16}\sin(-\ell''+2g'')-\frac{5}{2}\sin(2\ell''+2g'')\right. \\ &\quad \left.+\left(\frac{7}{6e}-\frac{397}{48}e\right)\sin(3\ell''+2g'')+\frac{17}{4}\sin(4\ell''+2g'')+\frac{169}{16}e\sin(5\ell''+2g'')\right\} \\ &\quad \left.+3\cos^2 i\left\{3e\sin\ell''+\frac{e}{2}\sin(\ell''+2g'')-\frac{1}{2}\sin(2\ell''+2g'')-\frac{7}{6}e\sin(3\ell''+2g'')\right\}\right] \end{aligned} \tag{388}$$

$$h = h'-\frac{3J_2R^2}{2a^2}\cos i\left[3e\sin\ell''+\frac{e}{2}\sin(\ell''+2g'')-\frac{1}{2}\sin(2\ell''+2g'')-\frac{7}{6}e\sin(3\ell''+2g'')\right] \tag{389}$$

2.2 Explicit な変換に基づく摂動論

2.2.1 Lie 級数

$x, y \to \xi, \eta$

$$\begin{cases} x_i = \frac{\partial S}{\partial y_i} \\ \xi_i = \frac{\partial S}{\partial \eta_i} \end{cases} \qquad S = S(\xi, y) = \sum \xi_i y_i + S_1 + S_2 + \cdots \tag{390}$$

すると，

$$\left.\begin{aligned} x_i &= \xi_i + \partial_{y_i} S_1 + \partial_{y_i} S_2 + \cdots \\ y_i &= \eta_i - \partial_{\xi_i} S_1 - \partial_{\xi_i} S_2 - \cdots \end{aligned}\right\} \tag{391}$$

このように implicit に表されるのではよく分かりにくい．そこで，Eq. (391) を解いて x, y を ξ, η で explicit に表そう．

$$\partial_{y_i} S_1(\xi, y)|_{y\to\eta} \equiv S_{1\eta_i}, \quad \partial_{\xi_i} S_1(\xi, y)|_{y\to\eta} \equiv S_{1\xi_i} \tag{392}$$

と書く．1 次までで

$$x_i = \xi_i + S_{1\eta_i}, \quad y_i = \eta_i - S_{1\xi_i} \tag{393}$$

2 次までで

$$x_i = \xi_i + S_{1\eta_i} - \sum_j S_{1\eta_i\eta_j} S_{1\xi_j} + S_{2\eta_i} \tag{394}$$
$$y_i = \eta_i - S_{1\xi_i} + \sum_j S_{1\xi_i\xi_j} S_{1\eta_j} - S_{2\xi_i} \tag{395}$$

すると，

$$\begin{aligned}
f(x,y) &= f(\xi,\eta)+\sum_i f_{\xi_i}\cdot(S_{1\eta_i}-\sum_j S_{1\eta_i\eta_j}S_{\xi_j}+S_{2\eta_i}) \\
&\quad +\sum_i f_{\eta_i}\cdot(-S_{1\xi_i}+\sum_j S_{1\xi_i\xi_j}S_{1\eta_i}-S_{2\xi_i}) \\
&\quad +\frac{1}{2}\sum_{ij} f_{\xi_i\xi_j}S_{1\eta_i}S_{1\eta_j}+\sum_{ij} f_{\xi_i\eta_j}S_{1\eta_i}(-S_{1\xi_j}) \\
&\quad +\frac{1}{2}\sum_{ij} f_{\eta_i\eta_j}(-S_{1\xi_i})(-S_{1\xi_j})+(3\text{ 次以上}) \\
&= f(\xi,\eta)+\sum(f_{\xi_i}S_{1\eta_i}-f_{\eta_i}S_{1\xi_i})+\sum(f_{\xi_i}S_{2\eta_i}-f_{\eta_i}S_{2\xi_i}) \\
&\quad -\sum\left[f_{\xi_i}(\sum_j S_{1\eta_j}S_{1\xi_j})_{\eta_i}-f_{\eta_i}(\sum_j S_{1\eta_j}S_{1\xi_j})_{\xi_i}\right] \\
&\quad +\sum f_{\xi_i}\sum S_{1\eta_j}S_{1\xi_j\eta_i}-\sum f_{\xi_i}\sum S_{1\eta_j}S_{1\xi_i\xi_j} \\
&\quad +\frac{1}{2}\sum(\sum f_{\xi_j}S_{1\eta_j})_{\xi_i}S_{1\eta_i}-\frac{1}{2}\sum\sum f_{\xi_j}S_{1\eta_j\xi_i}S_{1\eta_i} \\
&\quad -\frac{1}{2}\sum(\sum f_{\eta_j}S_{1\xi_j})_{\eta_i}S_{1\xi_i}-\frac{1}{2}\sum(\sum f_{\xi_j}S_{1\eta_j})_{\eta_i}S_{1\xi_i} \\
&\quad +\frac{1}{2}\sum(\sum f_{\eta_j}S_{1\xi_j})_{\eta_i}S_{1\eta_i}-\frac{1}{2}\sum\sum f_{\eta_j}S_{1\xi_j\eta_i}S_{1\xi_i} \qquad (396)\\
&\quad S_1(\xi,y)|_{y\to\eta}=S_1,\quad S_2(\xi,y)|_{y\to\eta}=S_2 \qquad (397)
\end{aligned}$$

$$f(x,y)=f(\xi,\eta)+\{f,S_1\}+\{f,S_2-\frac{1}{2}\sum S_{1\xi_i}S_{1\eta_i}\}+\frac{1}{2}\{\{f,S_1\},S_1\}+(3\text{ 次以上}) \qquad (398)$$

特に $f(x,y)=x$ として，

$$\begin{aligned}
x_i &= \xi_i+\{\xi_i,S_1\}+\{\xi_i,S_2-\frac{1}{2}\sum S_{1\xi_j}S_{1\eta_j}\}+\frac{1}{2}\{\{\xi_i,S_1\},S_1\} \\
&= \xi_i+S_{1\eta_i}+S_{2\eta_i}-\frac{1}{2}(\sum S_{1\xi_j S_{1\eta_j}})_{\eta_i}+\frac{1}{2}\{S_{1\eta_i},S_1\} \\
&\quad (\{S_{1\eta_i},S_1\}=\sum(S_{1\eta_i\xi_j}S_{1\eta_j}-S_{1\eta_i\eta_j}S_{1\xi_j})) \\
&= \xi_i+S_{1\eta_i}+S_{2\eta_i}-\sum S_{1\eta_i\eta_j}S_{1\xi_j} \qquad (399)
\end{aligned}$$

同様にして

$$y_i=\eta_i-S_{1\xi_i}-S_{2\xi_i}+\sum S_{1\xi_i\xi_j}S_{1\eta_j} \qquad (400)$$

(Von Zeipel) で $S_1(\xi,y),S_2(\xi,y)$ を得て，

$$\begin{cases}\tilde{S}_1\equiv S_1(\xi,y)|_{y\to\eta} \\ \tilde{S}_2\equiv S_2(\xi,y)-\frac{1}{2}\sum S_{1\xi_j}(\xi,y)S_{1y_j}(\xi,y)\Big|_{y\to\eta}\end{cases} \qquad (401)$$

とおけば，$\tilde{S}_1,\tilde{S}_2$ は正準不変．
変換：

$$f(x,y)=f(\xi,y)+\{f,\tilde{S}_1\}+\{f,\tilde{S}_2\}+\frac{1}{2}\{\{f,\tilde{S}_1\},\tilde{S}_1\}+(3\text{ 次以上}) \qquad (402)$$

特に，

$$x_i = \xi_i + \tilde{S}_{1\eta_i} + \tilde{S}_{2\eta_i} + \frac{1}{2}\{\tilde{S}_{1\eta_i}, \tilde{S}_1\} + \cdots \tag{403}$$

$$y_i = \eta_i - \tilde{S}_{1\xi_i} - \tilde{S}_{2\xi_i} - \frac{1}{2}\{\tilde{S}_{1\xi_1}, \tilde{S}_1\} - \cdots \tag{404}$$

2.2.2　展開定理 – Lie 級数

任意の母関数 $S(\xi,\eta)$

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\{f,S\}_n \tag{405}$$

ここで，

$$\{f,\tilde{S}\}_n = \{f,\tilde{S}_{n-1},\tilde{S}\}, \quad \{f,\tilde{S}\}_0 = f \tag{406}$$

このとき，

$$\begin{cases} x_i = \xi_i + \{\xi_i, S\} + \frac{1}{2}\{\{\xi_i, S\}, S\} + \cdots \\ y_i = \eta_i + \{\eta_i, S\} + \frac{1}{2}\{\{\eta_i, S\}, S\} + \cdots \end{cases} \tag{407}$$

Eq. (407) は正準変換.

2.2.3　人工衛星の運動

運動方程式

$$\frac{d}{dt}(L,G,H) = \frac{\partial F}{\partial(\ell,g,h)} \tag{408}$$

$$\frac{d}{dt}(\ell,g,h) = -\frac{\partial F}{\partial(L,G,H)} \tag{409}$$

$$F = \underbrace{\frac{\mu^2}{2L^2}}_{F_0(L)} + F_1(L,G,H,\ell,g,-) + F_2(L,G,H,\ell,g,-) \tag{410}$$

2 次までで，展開定理より，

$$F_0(L) = F_0(L') + \{F_0,S_1\} + \{F_0,S_2\} + \frac{1}{2}\{\{F_0,S_1\},S_1\} \tag{411}$$

$$F_1(L,G,H,\ell,g,-) = F_1(L',G',H',\ell',g',-) + \{F_1,S_1\} \tag{412}$$

$$F_2(L,G,H,\ell,g,-) = F_2(L',G',H',\ell',g',-) \tag{413}$$

エネルギー積分

$$\begin{aligned} & F_0(L) + F_1(L,G,H,\ell,g,-) + F_2(L,G,H,\ell,g,-) \\ = \ & F_0^*(L') + F_1^*(L',G',H',\ell',g',-) + F_2^*(L',G',H',\ell',g',-) \end{aligned} \tag{414}$$

代入して比較する．

$$F_0^* = F_0 \tag{415}$$

$$F_1^* = \{F_0, S_1\} + F_1 \tag{416}$$

$$F_2^* = \{F_0, S_2\} + \frac{1}{2}\{\{F_0, S_1\}, S_1\} + \{F_1, S_1\} + F_2 \tag{417}$$

$F_0 = F_0(L')$ より，$\{F_0, S_j\} = F_{0L'} S_{j\ell'}$ より，

(1 次) $$S_1 = -\frac{1}{F_{0L'}} \int (F_1 - F_1^*) d\ell' \tag{418}$$

$$F_1^* = \langle F_0 \rangle_{\ell'} \tag{419}$$

(2 次) $$S_2 = -\frac{1}{F_{0L'}} \int \left(\frac{1}{2}\{\{F_0, S_1\}, S_1\} + \{F_1, S_1\} + F_2 - F_2^* \right) d\ell' \tag{420}$$

$$F_2^* = \left\langle \frac{1}{2}\{\{F_0, S_1\}, S_1\} + \{F_0, S_2\} + F_2 \right\rangle_{\ell'} \tag{421}$$

勿論，この S_2 は Von Zeipel の S とは異なる．

2.2.4　運動方程式を解く

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \tag{422}$$

これを正準変換で解く $x, y \to \xi, \eta$

エネルギー積分

$$F_0(x,y) + F_1(x,y) + F_2(x,y) + \cdots = F_0^*(\xi,\eta) + F_1^*(\xi,\eta) + F_2^*(\xi,\eta) + \cdots \tag{423}$$

展開定理より，

$$\begin{cases} F_0(x,y) = F_0(\xi,\eta) + \{F_0, S_1\} + \{F_0, S_2\} + \{\{F_0, S_1\}, S_1\} \\ F_1(x,y) = F_0(\xi,\eta) + \{F_1, S_1\} \\ F_2(x,y) = F_2(\xi,\eta) \end{cases} \tag{424}$$

例えば，

$$F_0 = -\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{\mu}{\sqrt{x^2 + y^2 + z^2}} \tag{425}$$

これでは，

$$F_0^* = F_0 \tag{426}$$

$$F_1^* = F_1 - \frac{\mu \vec{r}}{r^3} \cdot \frac{\partial S_1}{\partial \dot{\vec{r}}} + \dot{\vec{r}} \cdot \frac{\partial S_1}{\partial \vec{r}} \quad \leftarrow \text{難} \tag{427}$$

そこで，補助方程式，

$$\frac{d\xi_i}{d\tau} = \frac{\partial F_0^*}{\partial \eta_i}, \quad \frac{d\eta_i}{d\tau} = -\frac{\partial F_0^*}{\partial \xi_i} \quad (\text{これは Unperturbed}) \tag{428}$$

よって解けて，

$$\xi_i = \xi_i(\tau + C_1, \cdots, C_{2n}), \quad \eta_i = \eta_i(\tau + C_1, \cdots, C_{2n}) \tag{429}$$

このとき $\{\quad\}$ は正準不変量だから

$$\{F_0, S_1\} = \{F_0^*, S_1\} = -\frac{\partial S_1}{\partial \tau} \tag{430}$$

ゆえに，

1 次の摂動

$$-\frac{\partial S_1}{\partial \tau} + F_1(\xi(\tau, C), \eta(\tau, C)) = F_1^* \tag{431}$$

$$F_1^* = \langle F_1 \rangle_\tau = F_1^*(\xi, \eta) \tag{432}$$

$$S_1 = \int (F_1 - F_1^*) d\tau = S_1(\xi, \eta) \tag{433}$$

と Eq. (429) を使って書ける．

2 次の摂動

$$F_2^* = \left\langle \frac{1}{2}\{\{F_0, S_1\}, S_1\} + \{F_1, S_1\} + F_2 \right\rangle_\tau = F_2^*(\xi, \eta) \tag{434}$$

$$S_2 = \int \left(\frac{1}{2}\{\{F_0, S_1\}, S_1\} + \{F_1, S_1\} + F_2 - F_2^* \right) d\tau = S_2(\xi, \eta) \tag{435}$$

新運動方程式は，

$$\frac{d\xi_i}{dt} = \frac{\partial F^*}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial F^*}{\partial \xi_i} \quad (i = 1, \cdots, n) \tag{436}$$

さて，

$$0 = \frac{dF^*}{d\tau} = -\{F_0^*.F^*\} = \{F^*, F_0^*\} = -\frac{dF_0^*}{dt} \tag{437}$$

新しい積分，

$$F_0^*(\xi, \eta) = \text{const.} \tag{438}$$

2.3 $e, i \sim 0$ の場合（人工衛星の運動続編）

2.3.1 対称性のある場合

1. 軸対称，赤道面対称

$$Y_{\ell m} \propto \delta_{m0} P_\ell \quad (\text{軸対称}) \tag{439}$$

$$Y_{\ell m} \propto \begin{cases} P_{2\ell} & (\text{Even}) \\ P_{2\ell+1} (= 0) & (\text{Odd}) \end{cases} \quad (\text{赤道面対称}) \tag{440}$$

明らかに，赤道面上を円運動するという解が存在する．

$$r = \text{const.}, \quad \dot{\theta} = \text{const.}$$

これは，

$$r = a = \text{const.}, \quad e = 0, \quad i = 0$$

と書いて良いだろうか？

2. 軸対称

$$r = \text{const.}, \quad \beta = \text{const.}, \quad \dot{\theta} = \text{const.} \tag{441}$$

運動方程式は，

$$\ddot{r} - r\dot{\beta}^2 - r\dot{\theta}^2\cos^2\beta = U_r \tag{442}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\beta}) + r\dot{\theta}^2\sin\beta\cos\beta = \frac{1}{r}U_\beta \tag{443}$$

$$\frac{1}{r\cos\beta}\frac{d}{dt}(r^2\dot{\theta}\cos^2\beta) = \frac{1}{r\cos\beta}U_\theta \tag{444}$$

$$U = \frac{\mu}{r}\left[1 - \sum_{k=2}\frac{J_k R^k}{r^k}P_k(\sin\beta)\right] \tag{445}$$

(定常解)
$r, \beta, \dot{\theta} = \text{const.}$

$$r\dot{\theta}^2\cos^2\beta = U_r = \frac{\mu}{r^2}\left[1 + \sum\frac{(k+1)J_k R^k}{r^k}P_k(\sin\beta)\right] \tag{446}$$

$$r^2\dot{\theta}^2\sin\beta = U_\beta = -\sum\frac{\mu J_k R^k}{r^{k+1}}P_k(\sin\beta) \tag{447}$$

$$r^2\dot{\theta}\cos^2\beta = H \quad (\text{const.}) \tag{448}$$

1. 赤道面対称
$P_{2\ell}$ のみ $P_{2\ell} \propto \sin\beta$．よって，$\beta = 0$ は満たされる．すると，

$$\dot{\theta}^2 = \frac{\mu}{r^3}\left[1 + \sum_{k=2}\frac{(2k+1)J_{2k}R^{2k}}{r^{2k}}P_{2k}(0)\right] \tag{449}$$

$$r^2\dot{\theta} = H \tag{450}$$

よって，

$$H^2 = \mu r\left[1 + \sum\frac{(2k+1)J_{2k}R^{2k}}{r^{2k}}P_{2k}(0)\right] \tag{451}$$

2. 軸対称
確かに H を与えれば $r, \beta, \dot{\theta}$ の方程式を解けば良い．
ところで i, e の意味は？

2.3.2 運動方程式

$$\frac{d}{dt}(L,G,H) = \frac{\partial F}{\partial(\ell,g,h)} \tag{452}$$

$$\frac{d}{dt}(\ell,g,h) = -\frac{\partial F}{\partial(L,G,H)} \tag{453}$$

$$F = \frac{\mu^2}{2L^2} + \sum_{k=2}\frac{\mu J_k R^k}{r^{k+1}}P_k(\sin\beta) \tag{454}$$

定常解を求めよう．まず $H = \mathrm{const.}$

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial F}{\partial \ell} \\ &= \sum(k+1)\frac{\mu J_k R^k}{r^{k+2}}P_k(\sin\beta)\frac{\partial r}{\partial \ell} - \sum\frac{\mu J_k R^k}{r^{k+1}}P_k(\sin\beta)\sin i\cos(f+g)\frac{\partial f}{\partial \ell} \\ &= 0 \end{aligned} \tag{455}$$

$$\begin{aligned}\frac{dG}{dt} &= \frac{\partial F}{\partial g} \\ &= -\sum\frac{\mu J_k R^k}{r^{k+1}}P_k(\sin\beta)\sin i\cos(f+g) = 0\end{aligned} \tag{456}$$

Eq. (456) より，

$$f + g = \pm\frac{\pi}{2} \quad (\sin i = 0 \text{ は残りの項を満足しない})$$

自動的に Eq. (455) より，

$$\frac{\partial r}{\partial \ell} = 0 \rightarrow \text{近（遠）地点} \quad \ell = 0 \ \text{ or } \ \pi$$

よって，

$$f = 0, \pi \rightarrow g = \pm\frac{\pi}{2} \tag{457}$$

1. $\ell = 0, g = \frac{\pi}{2}$
   ℓ, g, h は独立だから，

$$\frac{d\ell}{dt} = -\left(\frac{\partial F}{\partial L}\right)_{\ell=0,g=\frac{\pi}{2}} = -\frac{\partial}{\partial L}(F)_{\ell=0,g=\frac{\pi}{2}} \tag{458}$$

$$\frac{dg}{dt} = -\frac{\partial}{\partial G}(F) \tag{459}$$

$$\frac{dh}{dt} = -\frac{\partial}{\partial H}(F) \tag{460}$$

   ただし，

$$(F)_{\ell=0,g=\frac{\pi}{2}} = (F) \tag{461}$$

とした．$\ell=0, g=\frac{\pi}{2}$ より，

$$\sin\beta=\sin i, \quad r=a(1-e) \quad (\text{近地点}) \tag{462}$$

$$(F)=\frac{\mu}{2L^2}-\sum\frac{\mu J_k R^k}{[a(1-e)]^{k+1}} \tag{463}$$

さて，

$$\begin{aligned}\frac{\partial}{\partial L} &= \frac{\partial a}{\partial L}\frac{\partial}{\partial a}+\frac{\partial e}{\partial L}\frac{\partial}{\partial e}=\frac{2a}{L}\frac{\partial}{\partial a}+\frac{(1-e^2)}{eL}\frac{\partial}{\partial e} &(464)\\ \frac{\partial}{\partial G} &= -\frac{\sqrt{1-e^2}}{eL}\frac{\partial}{\partial e}+\frac{\cos i}{\sin i\sqrt{1-e^2}L}\frac{\partial}{\partial i} &(465)\\ \frac{\partial}{\partial H} &= -\frac{1}{\sin i\sqrt{1-e^2}L}\frac{\partial}{\partial i} &(466)\end{aligned}$$

よって，

$$\left(\frac{d\ell}{dt}\right)_{\ell=0,g=\frac{\pi}{2}}=-\frac{\partial}{\partial L}(F)=\frac{\mu^2}{L^2}+\sum\frac{(k+1)\mu J_k R^k}{[a(1-e)]^{k+2}}P_k(\sin i)\frac{a(1-e)^2}{eL}=0 \tag{467}$$

または[16]，

$$e=-\sum\frac{(k+1)J_k R^k}{[a(1-e)]^k}P_k(\sin i)\neq 0 \tag{468}$$

$$\begin{aligned}\left(\frac{dg}{dt}\right)_{\ell=0,g=\frac{\pi}{2}} &= -\frac{\partial}{\partial G}(F)\\ &= -\sum\frac{(k+1)\mu J_2 R^k}{[a(1-e)]^{k+2}}P_k(\sin i)\frac{a\sqrt{1-e^2}}{eL}\\ &\quad +\sum\frac{\mu J_k R^k}{[a(1-e)]^{k+1}}P_k(\sin i)\frac{\cos^2 i}{\sin i\sqrt{1-e^2}L}=0 \qquad (469)\end{aligned}$$

または，

$$\sum\frac{J_k R^k}{[a(1-e)]^{k+1}}\left[(k+1)P_k(\sin i)\frac{1+e}{e}-P_k(\sin i)\frac{\cos i}{\sin i}\right]=0 \tag{470}$$

ここで，

$$(x^2-1)P_k(x)=(k+1)P_{k+1}(x)-xP_k(x) \tag{471}$$

より，

$$\sum\frac{(k+1)J_k R^k}{[a(1-e)]^k}\left[\frac{P_k(\sin i)}{e}+\frac{P_{k+1}(\sin i)}{\sin i}\right]=0 \tag{472}$$

---

[16] $\frac{\mu}{L}=na$

よって，

$$\sin i = \sum \frac{(k+1)J_k R^k}{[a(1-e)]^k} P_{k+1}(\sin i) \tag{473}$$

特に $J_2, J_3 \gg J_4$ の時，

$$\sin i = \frac{3}{2} J_3 \left[\frac{R}{a(1-e)}\right]^3 + \cdots \tag{474}$$

$$e = \frac{3}{2} J_2 \left[\frac{R}{a(1-e)}\right]^2 \tag{475}$$

注意：

- $J_3 < 0 \cdots \ell = 0, g = \frac{\pi}{2}$ でよい．
- $J_2 > 0 \cdots \ell = \pi$ は解ではない．

$$\begin{aligned}
\left(\frac{dh}{dt}\right)_{\ell=0, g=\frac{\pi}{2}} &= -\frac{\partial}{\partial H}(F) \\
&= -\sum \frac{\mu J_k R^k}{[a(1-e)]^{k+1}} \frac{\cos i}{\sin i \sqrt{1-e^2} L} P_k(\sin i) \\
&= -\frac{na}{\cos i \sqrt{1-e^2}} \frac{1+e}{e} \sum \frac{(k+1)J_k R^k}{[a(1-e)]^k} P_k(\sin i) \\
&= \frac{na}{\cos i \sqrt{1-e^2}} \frac{1+e}{1-e}
\end{aligned} \tag{476}$$

よって，

$$h = \left[\frac{\frac{1+e}{1-e}}{\cos i \sqrt{1-e^2}} n\right] t + h_0 \tag{477}$$

すると，長半径 a

$$r = a(1-e)$$

円軌道の半径

$$a(1-e)\cos i \quad (i = -\beta)$$

しかし，おかしい点がある．

$$J_k \to 0 \Longrightarrow \mathrm{Kepler}(\dot{\ell} \neq 0)$$

ところがこの解を $J_k \to 0$ とすると，

$$\dot{h} \neq 0, \quad \dot{\ell} = 0$$

となってしまう．

2.3.3　赤道面内の運動

赤道面対称と考える[17]．F として J_2 までとる[18]．

$$\begin{aligned} \frac{d}{dt}(L,G) &= \frac{\partial F}{\partial(\ell,g)} && (478) \\ \frac{d}{dt}(\ell,\varpi) &= -\frac{\partial F}{\partial(L,G)} && (479) \\ F &= \frac{\mu^2}{2L^2}+\frac{\mu J_2 R^2}{2r^3} && (480) \end{aligned}$$

定常解は，

$$\begin{aligned} e_0 &= \frac{3}{2}J_2\left[\frac{R}{a(1-e_0)}\right]^2 && (481) \\ \ell_0 &= 0 && (482) \\ \varpi_0 &= \frac{\frac{1+e}{1-e}}{\sqrt{1-e^2}}nt+\varpi_0 && (483) \end{aligned}$$

一般解：2 個の第一積分が既に存在する．

$$\begin{aligned} G &= \text{const.} \Longrightarrow a(1-e^2)=\text{const.} && (484) \\ F &= \frac{\mu^2}{2L^2}+\frac{\mu J_2 R^2}{2r^3}=\text{const.} \Longrightarrow \frac{1}{a}+J_2R^2\frac{(1+e\cos f)^3}{a^3(1-e^2)^3}=\text{const.} && (485) \end{aligned}$$

変数変換

$$\gamma = J_2\left[\frac{R}{a(1-e)}\right]^2 \tag{486}$$

と置くと，

$$1-e^2+\gamma(1+e\cos f)^3=\text{const.} \tag{487}$$

すなわち，

$$\gamma(1-e\cos f)^3-e^2=\text{const.} \tag{488}$$

これは ℓ, e の関係である．
定常解では，

$$e_0=\frac{3}{2}\gamma(1+e_0^2) \tag{489}$$

---

[17] この場合昇降点 N が意味をなさない．したがって $g=\varpi$.
[18] $g=\varpi$.

γ	e_0	C_0	e_1	e_2
10^{-3}	0.0015	0.0010	0.0030	0.996
10^{-2}	0.0150	0.0102	0.030	0.960
10^{-1}	0.2251	0.1332	0.458	0.520
1/7	0.4514	0.2330	1.000	
1/6	1.0000	0.3333		

ここで，

$$\gamma(1+e_0^2)^2 \equiv C_0 \tag{490}$$

とする．

2.4　$e, i \approx 0$ の場合の一般解

Von Zeipel の方法は $\frac{1}{e}, \frac{1}{\sin i}$ を含むので，$e, i \approx 0$ の場合には適用出来ない．$S_1, S_1^*, F_1^{**}, F_2^{**}$ は使える．S_2 は使えない．
そこで，$e, i \approx 0$ でもはっきり定義できる量を独立変数にとろう．

$$\begin{cases} H \\ x_1 = \sqrt{2(L-G)}\cos\ell \\ x_2 = \sqrt{2(L-H)}\cos(\ell+g) \end{cases} \qquad \begin{cases} \lambda = \ell + g + h \\ y_1 = \sqrt{2(L-G)}\sin\ell \\ y_2 = \sqrt{2(L-H)}\sin(\ell+g) \end{cases} \tag{491}$$

変換は，

$$L = \frac{1}{2}(x_1^2+x_2^2+y_1^2+y_2^2)+H \tag{492}$$

$$G = \frac{1}{2}(x_2^2+y_2^2)+H \tag{493}$$

$$\ell = \tan^{-1}\frac{y_1}{x_1} \tag{494}$$

$$g = \tan^{-1}\frac{y_2}{x_2} - \tan^{-1}\frac{y_1}{x_1} \tag{495}$$

$$h = \lambda - \tan^{-1}\frac{y_2}{x_2} \tag{496}$$

さて，

$$\begin{aligned} S_1 = \frac{\mu J_2 R^2}{2na^3}\Bigg[& A\left(3e\sin\ell'' + \frac{9}{4}e^2\sin 2\ell''\right) \\ & + B\left\{-\frac{e}{2}\sin(\ell''+2g'') + \left(\frac{1}{2}-\frac{5}{4}e^2\right)\sin(2\ell''+2g'') \right. \\ & \left. + \frac{7}{6}e\sin(3\ell''+2g'') + \frac{17}{8}e^2\sin(4\ell''+2g'')\right\}\Bigg] + \mathcal{O}(e^3) \end{aligned} \tag{497}$$

$$S_1^* = -\frac{\mu J_2 R^2}{2na^3}\frac{e^2}{4}\frac{1-\frac{17}{2}\theta^2+\frac{15}{2}\theta^4}{1-5\theta^2}\sin(2g'') + \frac{\mu\frac{J_3}{J_2}R}{2na^2}e\sin i\cos g'' + \mathcal{O}(e^3) \tag{498}$$

$$
\begin{aligned}
F^{**} &= \frac{\mu^2}{2L''^2} + \frac{\mu J_2 R^2}{2a^3} A(1-e^2)^{-3/2} \\
&\quad + \frac{\mu J_2^2 R^4}{4a^5}\left[\frac{3}{8}(1-8\theta^2+19\theta^4) + \frac{3}{32}(7-82\theta^2+243\theta^4)e^2\right] + \mathcal{O}(e^3)
\end{aligned} \tag{499}
$$

ここで，

$$
\theta = \cos i = \frac{H''}{G''} \tag{500}
$$

とおいた．

解：

$$
L'', G'', H'' = \text{const.} \tag{501}
$$

$$
n = \frac{\mu^2}{L''^3}, \quad e = \left(1 - \frac{G''^2}{L''^2}\right)^{1/2}, \quad \theta = \cos i = \frac{H''}{G''} \tag{502}
$$

$$
\ell'' = -\left(\frac{\partial F^{**}}{\partial L''}\right) t + \ell_0 \tag{503}
$$

$$
g'' = -\left(\frac{\partial F^{**}}{\partial G''}\right) t + g_0 \tag{504}
$$

$$
h'' = -\left(\frac{\partial F^{**}}{\partial H''}\right) t + h_0 \tag{505}
$$

$$
\lambda'' = \ell'' + g'' + h'' = -\left(\frac{\partial F^{**}}{\partial L''} + \frac{\partial F^{**}}{\partial G''} + \frac{\partial F^{**}}{\partial H''}\right) t + \lambda_0 \tag{506}
$$

$$
\begin{array}{ccc}
\begin{pmatrix} L & \ell \\ G & g \\ H & h \end{pmatrix} & \rightarrow & \begin{pmatrix} H & \lambda \\ x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \\
\downarrow & & \downarrow \\
\begin{pmatrix} L' & \ell' \\ G' & g' \\ H' & h' \end{pmatrix} & \rightarrow & \begin{pmatrix} H' & \lambda' \\ x_1' & y_1' \\ x_2' & y_2' \end{pmatrix} \\
\downarrow & & \downarrow \\
\begin{pmatrix} L'' & \ell'' \\ G'' & g'' \\ H'' & h'' \end{pmatrix} & \rightarrow & \begin{pmatrix} H'' & \lambda'' \\ x_1'' & y_1'' \\ x_2'' & y_2'' \end{pmatrix}
\end{array}
$$

この図式は可換．ここで，

$$
x_1'' = \sqrt{2(L''-G'')}\cos\ell'' \tag{507}
$$

$$
x_2'' = \sqrt{2(L''-H'')}\cos(\ell''+g'') \tag{508}
$$

$$
y_1'' = \sqrt{2(L''-G'')}\sin\ell'' \tag{509}
$$

$$
y_2'' = \sqrt{2(L''-H'')}\sin(\ell''+g'') \tag{510}
$$

S_2 の時には { } の理論にしたがって，

$$S_2 = \tilde{S}_2 - \frac{1}{2}\sum \frac{\partial \tilde{S}_1}{\partial \xi_i}\frac{\partial \tilde{S}_2}{\partial \eta_i} \tag{511}$$

を計算する．ここで $\tilde{S}_2$ は真の Von Zeipel の正準不変量である．
e, i の 2 次の項まで取ると，

$$\begin{aligned} S_1 &= \frac{\mu^2 J_2 R^2}{2H''^3}\left(\frac{3y_1''}{\sqrt{H''}} + \frac{9}{2}\frac{x_1''y_1''}{H''} + \frac{3}{2}\frac{x_2''y_2''}{H''}\right) && (512)\\ S_1^* &= \frac{\mu\frac{J_3}{J_2}R}{2H''}\frac{x_1''x_2'' + y_1''y_2''}{H''} && (513)\end{aligned}$$

展開定理より[19]，

$$\begin{aligned} \lambda &= \lambda'' + \{\lambda'', S_1 + S_1^*\}\\ &= \lambda'' + \frac{\mu^2 J_2 R^2}{2H''^4}\left(\frac{21y_1''}{2\sqrt{H''}} + \frac{18x_1''y_1''}{H''} + \frac{6x_2''y_2''}{H''}\right) + \frac{\mu\frac{J_3}{J_2}R}{H''^2}\frac{x_1''x_2'' + y_1''y_2''}{H''}\\ &= \lambda'' + \frac{3J_2R^2}{2a^2}\left[\frac{7}{2}e\sin\ell'' + 3e^2\sin 2\ell'' + \sin^2 i \sin(2\ell'' + 2g'')\right] + \frac{\frac{J_3}{J_2}R}{a}s\sin i\cos g'' \end{aligned} \tag{514}$$

$$\begin{aligned} x_1 &= x_1'' + \{x_1'', S_1 + S_1^*\}\\ &= x_1'' + \frac{\mu^2 J_2 R^2}{2H''^3}\left(\frac{3}{\sqrt{H''}} + \frac{9x_1''}{2H''}\right) + \frac{\mu\frac{J_3}{J_2}R}{2H''}\frac{y_2''}{H''}\\ &= x_1'' + \frac{3nJ_2R^2}{2\sqrt{H''}}\left(1 + \frac{3}{2}e\cos\ell''\right) + \frac{na\frac{J_3}{J_2}R}{2\sqrt{H''}}\sin i\sin(\ell'' + g'' && (515)\\ y_1 &= y_1'' + \{y_1'', S_1 + S_1^*\}\\ &= y_1'' - \frac{3nJ_2R^2}{2\sqrt{H''}}\frac{3}{2}e\sin\ell'' - \frac{na\frac{J_3}{J_2}R}{2\sqrt{H''}}\sin i\cos(\ell'' + g'') && (516)\\ x_2 &= x_2'' + \frac{3nJ_2R^2}{2\sqrt{H''}}\frac{1}{4}\sin i\cos(\ell'' + g'') + \frac{na\frac{J_3}{J_2}R}{2\sqrt{H''}}e\sin\ell'' && (517)\\ y_2 &= y_2'' - \underbrace{\frac{3nJ_2R^2}{2\sqrt{H''}}}_{U}\frac{1}{4}\sin i\sin(\ell'' + g'') - \underbrace{\frac{na\frac{J_3}{J_2}R}{2\sqrt{H''}}}_{V}e\cos\ell'' && (518)\end{aligned}$$

そして，

$$\ell = \tan^{-1}\left[\frac{y_1'' - U\frac{3}{2}e\sin\ell'' - V\sin i\cos(\ell'' + g'')}{x_1'' + U\left(1 + \frac{3}{2}e\cos\ell''\right)V\sin i\sin(\ell'' + g'')}\right] \tag{519}$$

---

[19]

$$\{\lambda, S\} = -\frac{\partial S}{\partial H}, \quad \{\xi, S\} = -\frac{\partial S}{\partial \xi}, \quad \{\eta, S\} = -\frac{\partial S}{\partial \eta}$$

である．

代入して，

$$\ell = \tan^{-1}\left[\frac{e\sin\ell'' - \tilde{U}\frac{3}{2}e\sin\ell'' - \tilde{V}\sin i\cos(\ell''+g'')}{e\cos\ell'' + \tilde{U}\left(1+\frac{3}{2}e\cos\ell''\right) + \tilde{V}\sin i\sin(\ell''+g'')}\right] \tag{520}$$

$$\ell + g = \tan^{-1}\left[\frac{\sin i\sin(\ell''+g'') - \tilde{U}\frac{1}{4}\sin i\sin(\ell''+g'') - \tilde{V}e\cos\ell''}{\sin i\cos(\ell''+g'') + \tilde{U}\frac{1}{4}\sin i\cos(\ell''+g'') + \tilde{V}e\sin\ell''}\right] \tag{521}$$

$$h = \lambda - (\ell + g) \tag{522}$$

ここで，

$$\tilde{U} \equiv \frac{3J_2R^2}{2a^2}, \quad \tilde{V} \equiv \frac{\frac{J_3}{J_2}R}{2a} \tag{523}$$

ここで初めて，ℓ'', g'', h'' は長年摂動の意味ではないことが分かった．摂動がないと，$\tilde{U}, \tilde{V} = 0 (J_2 = J_3 = 0)$.

$$\ell = \ell'' = nt + \ell_0, \quad \ell + g = \ell'' + g'' = nt + \ell_0 + g_0 \tag{524}$$

となり，これは Kepler 運動．
6 個の積分定数：$L'', G'', H'', \ell_0, g_0, h_0$

1. 定常解 $L'' = G'' = H''$ (i, e $e(L'', G'', H'') = i(L'', G'', H'') = 0$)

$$x_1 = U = \text{const.}, \quad x_2 = 0, \quad y_1 = 0, \quad y_2 = 0 \tag{525}$$

   よって，

$$\ell = 0, \quad G = H = H'', \quad L = \frac{1}{2}x_1^2 + H \tag{526}$$

   これより[20]，

$$i_{\rm osc} = 0, \quad e_{\rm osc} = \frac{x_1}{\sqrt{H''}} = \tilde{U} = \frac{3J_2R^2}{2a^2} \tag{527}$$

2. 赤道面内の運動 ($J_3 = 0$)

$$J_3 = 0, \quad G'' = H'' \Rightarrow x_2'' = y_2'' = 0 \Rightarrow x_2 = y_2 = 0 \Rightarrow i_{\rm osc} = 0 \tag{528}$$

---

[20]

$$\cos i = \frac{H''}{G''}, \quad \cos i_{\rm osc} = \frac{H}{G}$$

osc：osculating (接触要素)

よって赤道面内の運動.

解：[21]

$$\frac{x_1}{\sqrt{H''}} = \underbrace{\frac{3J_2R^2}{2a^2}}_{e_0} + \left(1 + \frac{9J_2R^2}{4a^2}\right) e\cos\ell'' \tag{529}$$

$$\frac{y_1}{\sqrt{H''}} = \left(1 - \frac{9J_1R^2}{4a^2}\right) e\sin\ell'' \tag{530}$$

2 乗して加えると[22]，

$$\begin{aligned}\frac{2(L-G)}{H''} &= \frac{x_1^2 + y_1^2}{H''} \\ &= e_0^2 + (2+3e_0)e_0 e\cos\ell'' + \left(1 + \frac{9}{4}e_0^2\right)e^2 + 3e_0e^2\cos 2\ell''\end{aligned} \tag{531}$$

$$\begin{aligned}e_{\rm osc}^2 &= 1 - \frac{G^2}{L^2} \\ &= 1 - \left[\frac{H''}{\frac{1}{2}(x_1^2+y_1^2) + H''}\right]^2 \simeq \frac{x_1^2+y_1^2}{H''} \\ &= e^2 + e_0^2 + \frac{9}{4}e^2e_0^2 + (2+3e_0)e_0e\cos\ell'' + 3e_0e^2\cos 2\ell'' \end{aligned} \tag{532}$$

$$\ell = \tan^{-1}\left[\frac{\left(1 - \frac{3}{2}e_0\right)e\sin\ell''}{e_0 + \left(1 + \frac{3}{2}e_0\right)e\cos\ell''}\right] \tag{533}$$

Check：

$e = 0$; $e_{\rm osc} = e_0, \quad \ell = 0$

$e \neq 0$; 秤動の条件 (Libration) （ℓの範囲に制限がある）

$$e_{\rm cr} = \frac{e_0}{1 + \frac{3}{2}e_0} \simeq e_0, \quad e_0 \sim J_2 \sim 10^{-3} \ll 1$$

- $e < e_{\rm cr}$：秤動の条件
- $e > e_{\rm cr}$：周動 (Circulation), $\ell \to \infty$ が存在

$e = e_0$

$$e_{\rm osc}^2 \simeq 2e_0^2(1 + \cos\ell'') \tag{534}$$

$$= 4e_0^2\cos^2\frac{\ell''}{2} \tag{535}$$

---

[21]

$$x_1'' = e\cos\ell''\sqrt{H''}, \quad y_1'' = e\cos\ell''\sqrt{H''}$$

[22]

$$e^2 \equiv 1 - \frac{G''^2}{L''^2}$$

$$e_{\mathrm{osc}} = 2e_0 \left| \cos \frac{\ell''}{2} \right| \tag{536}$$

$$\ell \simeq \tan^{-1} \frac{\sin \ell''}{1 + \cos \ell''} \Rightarrow \ell \simeq \frac{\ell''}{2} \tag{537}$$

$$e_{\mathrm{osc}} = 2e_0 |\cos \ell| \tag{538}$$

$e > e_{\mathrm{cr}}$；周動
このとき，ℓ の平均運動は ℓ'' の平均運動と同じ．

$$\langle \ell \rangle = \langle \ell'' \rangle \tag{539}$$

3. $J_3 \neq 0, G'' = H''$

$$x_2'' = y_2'' = 0, \quad x_2 = y_2 = 0, \quad i = 0 \tag{540}$$

$$\left. \begin{array}{l} \frac{x_2}{\sqrt{H''}} = -\frac{\frac{J_3}{J_2}R}{2a} e \sin \ell'' \\ \frac{y_2}{\sqrt{H''}} = \frac{\frac{J_3}{J_2}R}{2a} \cos \ell'' \end{array} \right\} \Rightarrow \frac{x_2^2 + y_2^2}{H''} = \left(\frac{\frac{J_3}{J_2}R}{2a} e \right)^2 \tag{541}$$

$$\begin{aligned}
\sin^2 i_{\mathrm{osc}} &= 1 - \left[\frac{H''}{\frac{1}{2}(x_2^2 + y_2^2) + H''} \right]^2 \simeq \frac{x_2^2 + y_2^2}{H''} && (542)\\
\sin i_{\mathrm{osc}} &= \frac{\frac{J_3}{J_2} Re}{2a} = \mathrm{const.} && (543)\\
\ell + g &= \tan^{-1}(-\mathrm{cet}\ell'') = \begin{cases} \ell'' + \frac{\pi}{2} & (J_3 > 0) \\ \ell'' - \frac{\pi}{2} & (J_3 < 0) \end{cases} && (544)\\
h &= \lambda - (\ell + g) = \lambda \mp \frac{\pi}{2} && (545)\\
\lambda &= \lambda'' + \frac{3J_2R}{2a^2} \left(\frac{7}{2} e \sin \ell'' + 3e^2 \sin 2\ell'' \right) \\
h &= g'' + h'' \mp \frac{\pi}{2} + \text{周期項} && (546)
\end{aligned}$$

- 秤動の時：$\langle \ell \rangle = 0$ より，$\langle g \rangle = \ell'' \pm \frac{pi}{2}$
- 周動の時：$\langle \ell \rangle = \ell''$ より，$\langle g \rangle = \pm \frac{\pi}{2}$

4. $J_3 \neq 0, L'' = G''$

$$e = 0, \quad x_1'' + y_1'' = 0 \tag{547}$$

$$\lambda = \lambda'' + \frac{3J_2R^2}{2a^2}\sin^2 i \sin(2\ell'' + 2g'') \tag{548}$$

$$\frac{x_1}{\sqrt{H''}} = e_0 + \frac{\frac{J_3}{J_2}R}{2a}\sin i \sin(\ell'' + g'') \tag{549}$$

$$\frac{y_1}{\sqrt{H''}} = \frac{\frac{J_3}{J_2}R}{2a}\sin i \cos(\ell'' + g'') \tag{550}$$

$$\frac{x_2}{\sqrt{H''}} = \left(1 + \frac{e_0}{4}\right)\sin i \cos(\ell'' + g'') \tag{551}$$

$$\frac{y_2}{\sqrt{H''}} = \left(1 - \frac{e_0}{4}\right)\sin i \sin(\ell'' + g'') \tag{552}$$

$$\ell = \tan^{-1}\left[\frac{\frac{\frac{J_3}{J_2}R}{2a}\sin i \cos(\ell'' + g'')}{e_0 - \frac{\frac{J_3}{J_2}R}{2a}\sin i \sin(\ell'' + g'')}\right] \tag{553}$$

$$\ell + g = \tan^{-1}\left[\frac{\left(1 - \frac{e_0}{4}\right)\cos(\ell'' + g'')}{\left(1 + \frac{e_0}{4}\right)\sin(\ell'' + g'')}\right] \tag{554}$$

$$\sin^2 i_{\rm osc} \equiv 1 - \left[\frac{H''}{\frac{1}{2}(x_1^2 + y_1^2) + H''}\right]^2 \tag{555}$$

よって，

$$\sin i < \sin i_0 \equiv \frac{2ae_0}{\frac{J_3}{J_2}R} \tag{556}$$

なら ℓ は秤動．$\ell + g$ は常に周動（平均運動は，$\ell'' + g''$ のそれと同じ）．
$\to h$ は常に周動（平均運動は h'' のそれと同じ）．

ドロネー変数の長年項の表

	1.	3. $i = 0$		
	$e = i = 0$	$e < e_0$	$e = e_0$	$e > e_0$
ℓ	$(\ell \simeq 0)$	L	$\frac{\ell''}{2}$	ℓ''
g	–	ℓ''	$\frac{\ell''}{2}$	L
h	–	$g'' + h''$	$g'' + h''$	$g'' + h''$
$\varpi = g + h$	$\ell'' + g'' + h''$	$\ell'' + g'' + h''$	$\frac{\ell''}{2} + g'' + h''$	$g'' + h''$
cf.	(平衡解) $e_{\rm osc} = e$ $i_{\rm osc} = 0$	$e_0 = \frac{3J_2R^2}{2a^2}$ $\sin i_{\rm osc} = \frac{\frac{J_3}{J_2}Re}{2a}$		

Table 1: $i_{\rm osc} = 0$ だから g, h が定義できない．

	4. $e=0$			2. $J_3=0, i=0$		
	$i<i_0$	$i=i_0$	$i>i_0$	$e<e_0$	$e=e_0$	$e>e_0$
ℓ	L	$\frac{\ell''+g''}{2}$	$\ell''+g''$	L	$\frac{\ell''}{2}$	ℓ''
g	$\ell''+g''$	$\frac{\ell''+g''}{2}$	L	$-$	$-$	$-$
h	h''	h''	h''	$-$	$-$	$-$
$\varpi=g+h$	$\ell''+g''+h''$	$\frac{\ell''+g''}{2}+h''$	h''	$\ell''+g''+h''$	$\frac{\ell''}{2}+g''+h''$	$g''+h''$
cf.	$\sin i_0=\frac{2ae_0}{\frac{J_3}{J_2}R}$			$i_{\rm OSC}=0$		

Table 2: $i_{\rm OSC}=0$ だから g, h が定義できない.

3 非保存系の摂動論

正準形式が使えないときを扱う．

$$\frac{dz_j}{dt} = Z_j(z_1, \cdots, z_n) \quad (j = 1 \sim n) \tag{557}$$

しかし，これは Hamiltonian System に書ける．

$$F = \sum_{j=1} \phi_j Z_j(z) \tag{558}$$

とおき，

$$\frac{dz_j}{dt} = \frac{\partial F}{\partial \phi_j}, \quad \frac{d\phi_j}{dt} = -\frac{\partial F}{\partial z_j} \tag{559}$$

とすればよい．
正準変換，

$$z, \phi \longrightarrow \zeta, \pi \tag{560}$$

母関数，

$$\left.\begin{array}{l} S(\zeta,\pi) = S_1(\zeta,\pi) + S_2(\zeta,\pi) + \cdots \\ S_1(\zeta,\pi) = \sum \pi_j T_j^{(1)}(\zeta) \\ S_2(\zeta,\pi) = \sum \pi_j T_j^{(2)}(\zeta) \end{array}\right\} \tag{561}$$

さて，

$$Z_j(z) = Z_j^{(0)}(z) + Z_j^{(1)}(z) + \cdots \tag{562}$$

として，新しい Hamiltonian F^*，

$$\left.\begin{array}{l} F^*(\zeta,\pi) = F_0^*(\zeta,\pi) + F_1^*(\zeta,\pi) + \cdots \\ F_0^*(\zeta,\pi) = \sum \pi_j Z^{*(0)}(\zeta) \\ F_1^*(\zeta,\pi) = \sum \pi_j Z^{*(1)}(\zeta) \end{array}\right\} \tag{563}$$

変換：

$$z = \zeta + \{\zeta, S_1\} + \{\zeta, S_2\} + \frac{1}{2}\{\{\zeta, S_1\}, S_1\} + \cdots \tag{564}$$

$$\phi = \pi + \{\pi, S_1\} + \{\pi, S_2\} + \frac{1}{2}\{\{\pi, S_1\}, S_1\} + \cdots \tag{565}$$

$$Z(z) = Z(\zeta) + \{Z, S_1\} + \{Z, S_2\} + \frac{1}{2}\{\{Z, S_1\}, S_1\} + \cdots \tag{566}$$

さて，

$$\{\zeta, A\} = \frac{\partial}{\partial \pi}, \quad \{T(\zeta), A\} = T\frac{\partial}{\partial \pi} \tag{567}$$

$$z_j = \zeta_j + T_j^{(1)} + T_j^{(2)} + \frac{1}{2} T_i^{(1)} \frac{\partial T_j^{(1)}}{\partial \pi_i} + \cdots \tag{568}$$

$$\phi_j = \pi_j - \pi_i \frac{\partial T_i^{(1)}}{\partial \zeta_j} - \pi_i \frac{\partial T_i^{(2)}}{\partial \zeta_j} - \frac{1}{2} \left\{ \pi_i \frac{\partial T_i^{(1)}}{\partial \pi_j}, S_1 \right\} + \cdots \tag{569}$$

ここで,

$$\begin{aligned} \{\quad,\quad\}_j &= \pi_i (\partial_j \partial_k T^i) T^k - \pi_k (\partial_i T^k)(\partial_j T^i) \\ &= \pi_k (\partial_j \partial_i T^k) T^i - \pi_k (\partial_i T^k)(\partial_j T^i) \end{aligned} \tag{570}$$

$$\begin{aligned} \sum \phi_j Z_j^{(0)}(z) &= \sum \pi_j Z_i^{(0)} + \{\sum \pi_j Z_j^0, \sum \pi_j Z_j^{(1)}\} + \cdots \\ &= \pi_j Z_j^{(0)} + \pi_j \left(\partial_i Z_j^{(0)} T_i^{(1)} - Z_i^{(0)} \partial_i T_j^{(1)} \right) + \pi_j \left(\partial_i Z_j^{(0)} T_i^{(2)} - Z_i^{(0)} \partial_i T_j^{(2)} \right) \\ &\quad + \frac{1}{2} \left\{ \pi_j \left(\partial_i Z_j^{(0)} T_i^{(1)} - Z_i^{(0)} \partial_i T_j^{(1)} \right), \pi_j T_j \right\} + \cdots \\ &\quad + \pi_j Z_j^{(1)} + \pi_j (\partial_i Z_j^{(1)} T_i^{(1)} - Z_i^{(1)} \partial_i T_j^{(1)}) + \cdots \\ &\quad + \pi_j Z_J^{(2)} + \cdots \\ &= \pi_j Z_j^{*(0)} + \pi_j Z_j^{*(1)} + \pi_j Z_j^{*(2)} + \cdots \end{aligned} \tag{571}$$

新しい運動方程式：

$$\frac{d\zeta_j}{dt} = \frac{\partial}{\partial \pi_j} \sum \pi_k Z_k^* = Z_j^*(\zeta), \quad \frac{d\pi_j}{dt} = -\frac{\partial}{\partial \zeta_j} \sum \pi_k Z_k^* = -\sum \pi_k \frac{\partial Z_k^*}{\partial \zeta_j} \tag{572}$$

展開定理：

$$f = f(z, \phi) = \sum \phi_j f_j(z) \tag{573}$$

Lie Series

$$f(z, \phi) = f(\zeta, \pi) + \{f, S\} + \frac{1}{2} \{\{f, S\}, S\} + \cdots \tag{574}$$

ところで,

$$\begin{aligned} \{f, S\} &= \{\sum \pi_j f_j(\zeta), \sum \pi_j T_j\} \\ &= \sum_k \left[\left(\frac{\partial}{\partial \zeta_k} \sum \pi_j f_j \right) \left(\frac{\partial}{\partial \pi_k} \sum \pi_i T_i \right) - \left(\frac{\partial}{\partial \pi_k} \sum \pi_j f_j \right) \left(\frac{\partial}{\partial \zeta_k} \sum \pi_i T_i \right) \right] \\ &= \sum_k \left[\sum_j \pi_j \frac{\partial f_j}{\partial \zeta_k} \cdot T_k - f_k \cdot \sum_i \pi_i \frac{\partial T_i}{\partial \zeta_k} \right] \\ &= \pi_j \left(\frac{\partial f_j}{\partial \zeta_k} T_k - f_k \frac{\partial T_j}{\partial \zeta_k} \right) \\ &\equiv \pi_j [f, T]_j \end{aligned} \tag{575}$$

とおくと[23]，

$$\{\{f,S\},S\} = \{\pi_j[f,T]_j, \pi_j T_j\} = \pi_j[[f,T],T]_j \tag{576}$$

よって，

$$f(z,\phi) = \pi_j \left\{ f_j(\zeta) + [f,T]_j + \frac{1}{2}[[f,T],T]_j + \cdots \right\} \tag{577}$$

特に f が ϕ によならければ，

$$f(z) = f(\zeta) + \{f, \pi_j T_j\} + \frac{1}{2}\{\{f,\pi_j T_j\}, \pi_k T_k\} + \cdots \tag{578}$$

$$\{f, \pi_j T_j\} = \frac{\partial f}{\partial \zeta_k} T_k \equiv D_T f \tag{579}$$

$$\{\{f,\pi_j T_j\}, \pi_k T_k\} = D_T(D_T f) \equiv D^2 f \tag{580}$$

$$D_T \equiv T_k \frac{\partial}{\partial \zeta_k} \tag{581}$$

Lie Derivative[24]

$$f(z) = f(\zeta) + \sum \frac{1}{n!} D_T^n f \tag{582}$$

エネルギー積分，

$$F = F^* \tag{583}$$

より，

$$\phi_j Z_j^{(0)}(z) + \phi_j Z_j^{(1)}(z) + \phi_j Z_j^{(2)}(z) + \cdots = \phi_j Z_j^{*(0)}(\zeta) + \phi_j Z_j^{*(1)}(\zeta) + \phi_j Z_j^{*(2)}(\zeta) + \cdots \tag{584}$$

展開定理より，

$$\phi_j Z_j^{(0)}(z) = \pi_j Z_j^{(0)} + \pi_j [Z^{(0)}, T^{(1)} + T^{(2)}]_j + \frac{1}{2}\pi_j [[Z^{(0)}, T^{(1)}], T^{(1)}]_j + \cdots \tag{585}$$

$$\phi_j Z_j^{(1)}(z) = \pi_j Z^{(1)} + \pi_j [Z^{(1)}, T^{(1)}] + \cdots \tag{586}$$

$$\phi_j Z_j^{(2)}(z) = \phi_j Z^{(2)} + \cdots \tag{587}$$

比較して，

$$(0\,次) \qquad \pi_j Z_j^{(0)}(z) = \pi_j Z^{*(0)} \Rightarrow Z_j^{(0)} = Z^{*(0)} \quad (j = 1 \sim n) \tag{588}$$

$$(1\,次) \qquad \pi_j [Z^{(0)}, T^{(1)}]_j + \pi_j Z_j^{(1)} = \pi_j Z_j^{*(1)} \Rightarrow Z_j^{*(1)} = [Z^{(0)}, T^{(1)}]_j + Z_j^{(1)} \tag{589}$$

$$(2\,次) \qquad Z_j^{*(2)} = Z_j^{(2)} + [Z^{(0)}, T^{(1)}]_j + [Z^{(1)}, T^{(1)}]_j + \frac{1}{2}[[Z^{(0)}, T^{(1)}], T^{(1)}]_j + \cdots \tag{590}$$

23

$$[A,B]_j = \frac{\partial A_j}{\partial \zeta_k} B_k - A_k \frac{\partial B_j}{\partial \zeta_k}$$

24

$$D_S f \equiv \{f.S\}$$

これも Lie Derivative.

ここで，

$$[Z^{(0)}, T^{(1)}] = Z^{*(1)} - Z^{(1)} \tag{591}$$

より，

$$Z_j^{*(2)} = Z_j^{(2)} + [Z^{(0)}, T^{(2)}]_j + \frac{1}{2}[Z^{*(1)} + Z^{(1)}, T^1]_j \tag{592}$$

各段階で n 個の方程式から n 個の T と n 個の Z^* を決めなければならない．
さて，

$$\{F_0, S_1\} + F_1 = F_1^* \tag{593}$$

これを解くのに，τ を導入して，

$$\{F_0, A\} = -\frac{dA}{d\tau}, \quad \frac{d\xi}{d\tau} = \frac{\partial F_0}{\partial \eta}, \quad \frac{d\eta}{d\tau} = -\frac{\partial F_0}{\partial \xi} \tag{594}$$

として解いた．そこで同じ様にやる．
補助方程式：

$$\frac{d\zeta_j}{d\tau} = Z_j^{*(0)}(\zeta) \tag{595}$$

解：

$$\zeta_j = \zeta(\tau + C_1, C_2, \cdots, C_n) \tag{596}$$

すると，

$$\begin{aligned}
[Z^{(0)}, A]_j &= \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} A_k - Z_k \frac{\partial A_j}{\partial \zeta_k} \\
&= \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} A_k - \frac{d\zeta_k}{d\tau}\frac{\partial A_j}{\partial \zeta_k} \\
&= \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} A_k - \frac{dA_j}{d\tau}
\end{aligned} \tag{597}$$

よって，

$$[Z^{(0)}, A]_j = -\frac{dA_j}{d\tau} + A_k \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} \tag{598}$$

1 次で，

$$-\frac{dT_j^{(1)}}{d\tau} + T_k^{(1)} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} + Z_j^{(1)} = Z_j^{*(1)} \tag{599}$$

これは var. coeff linear D. eq.s

平均値法の原理，

$$
\begin{aligned}
0 &= \frac{dF^*}{d\tau} = \{F^*, F_0^*\} = -\{F_0^*, F^*\} \\
&= -\{\pi_j Z_j^{*(0)}, \pi_k Z_k^*\} = -\pi_j [Z^{*(0)}, Z^*]_j
\end{aligned} \tag{600}
$$

π_j によらないから，

$$
[Z^{*(0)}, Z^*]_j = 0 \quad (j = 1 \sim n) \tag{601}
$$

Order で比較して，

$$
[Z^{*(0)}, Z^{*(\ell)}]_j = 0 \quad \ell = 1, 2, \cdots \tag{602}
$$

これを満たさなければならない[25]．これが満たされると，

$$
\frac{dZ_j^{*(0)}}{dt} = \frac{\partial Z_j^{*(0)}}{\partial \zeta_k}\frac{d\zeta_k}{dt} = \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} Z_k^* = \frac{\partial Z_j^{*(0)}}{\partial \zeta_k}\frac{d\zeta_k}{d\tau} \tag{603}
$$

よって，

$$
\frac{dZ_j^{*(0)}}{dt} = \frac{dZ_j^*}{d\tau} (\neq 0 \quad \text{in general}) \quad j = 1 \sim n \tag{604}
$$

また，

$$
\frac{dZ_j^{*(\ell)}}{d\tau} = \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} Z_j^{*(\ell)} \quad \ell = 1, 2, \cdots \tag{605}
$$

- ex. 1

  $$
  \ddot{x} + x + \epsilon \dot{x} = 0, \quad \epsilon \ll 1
  $$

  解：

  $$
  x = Ce^{-\frac{\epsilon}{2}t} \cos\left(\sqrt{1 - \frac{\epsilon^2}{4}} t + C'\right)
  $$

  1. 不用意な解法

  $$
  (0\text{次}) \quad \ddot{x}_0 + x_0 = 0 \rightarrow x_0 = cos(t + C') \tag{606}
  $$

  $$
  \begin{aligned}
  (1\text{次}) \quad & \ddot{x}_1 + x_1 = -\epsilon \dot{x}_0 = \epsilon \sin(t + C') \\
  & \rightarrow x_1 = C\cos(t + C') + \frac{\epsilon C}{2} t \cos(t + C') + \mathcal{O}(\epsilon)
  \end{aligned} \tag{607}
  $$

  $$
  \begin{aligned}
  (2\text{次}) \quad & \ddot{x}_2 + x_2 = -\epsilon \dot{x}_1 = \epsilon C \sin(t + C') - \frac{\epsilon^2 C}{2}\cos(t + C') + \frac{\epsilon^2 C}{2} t \sin(t + C') \\
  & \rightarrow x_2 = C\cos(t + C') - \frac{\epsilon C}{2} t \cos(t + C') + \frac{\epsilon^2 C}{8} t \sin(t + C') \\
  & \quad + \frac{\epsilon^2 C}{8} t^2 \cos(t + C') + \mathcal{O}(\epsilon^2)
  \end{aligned} \tag{608}
  $$

25

$$
[A, B] = B\frac{\partial A}{d\zeta} - A\frac{\partial B}{\partial \zeta}
$$

しかし，これは近付いては行くが，途中 では $t \nearrow$ で発散する（平均法を使わないから）.

2. 良い解法

$$x = z_1, \quad \frac{dz_1}{dt} = z_2, \quad \frac{dz_2}{dt} = -z_1 - \epsilon z_2$$
$$Z_1^{(0)}(z) = z_2, \quad Z_1^{(1)}(z) = 0, \quad Z_2^{(0)}(z) = -z_1, \quad Z_2^{(1)}(z) = -\epsilon z_2$$

変換

$$z_1, z_2 \to \zeta_1, \zeta_2, \quad T_j(\zeta) = T_j^{(1)} + T_j^2 + \cdots \quad j = 1, 2 \tag{609}$$

補助方程式

$$\frac{d\zeta_1}{d\tau} = \zeta_2, \quad \frac{d\zeta_2}{d\tau} = -\zeta_1 \Rightarrow \begin{cases} \zeta_1 = C\cos(\tau + C') \\ \zeta_2 = -C\sin(\tau + C') \end{cases} \tag{610}$$

ここの C, C' は τ に対する積分定数.

(0 次) $$Z_j^{*(0)} = Z_j^{(0)} \tag{611}$$

(1 次) $$\begin{cases} -\frac{dT_1^{(1)}}{d\tau} + T_2^{(1)} = Z_1^{*(1)} \\ -\frac{dT_2^{(1)}}{d\tau} - T_1^{(1)} - \epsilon\zeta_2 = Z_2^{*(1)} \end{cases} \tag{612}$$

平均値法の条件

$$\begin{cases} \frac{dZ_1^{*(1)}}{d\tau} = Z_2^{*(1)} \\ \frac{dZ_2^{*(1)}}{d\tau} = -Z_1^{*(1)} \end{cases} \tag{613}$$

これより，

$$\frac{d^2T_1^{(1)}}{d\tau^2} = \frac{dT_2^{(1)}}{d\tau} - \frac{dZ_1^{*(1)}}{d\tau} = -T_1^{(1)} - \epsilon\zeta_2 - 2Z_2^{*(1)}$$
$$\frac{d^2T_1^{(1)}}{d\tau^2} + T_1^{(1)} = \underbrace{-\epsilon\zeta_2}_{+\epsilon C\sin(\tau+C')} - 2Z_2^{*(1)} \tag{614}$$

ここで，

$$Z_2^{*(1)} = -\frac{\epsilon}{2}\zeta_2, \quad T_1^{(2)} = 0 \tag{615}$$

とおけばよい.

$$T_2^{(1)} = Z_1^{*(1)} = +\frac{\epsilon}{2}\frac{d\zeta_2}{d\tau} = -\frac{\epsilon}{2}\zeta_2 \tag{616}$$

(2 次) (617)

$$-\frac{dT_1^{(2)}}{d\tau} + T_2^{(2)} + \frac{1}{2}[Z^{(1)} + Z^{*(1)}, T^{(1)}]_1 = Z_1^{*(2)} \tag{618}$$
$$-\frac{dT_2^{(2)}}{d\tau} - T_1^{(2)} + \frac{1}{2}[Z^{(1)} + Z^{*(1)}, T^{(1)}]_2 = Z_2^{*(2)} \tag{619}$$

ここで，

$$[\quad,\quad]_1 = 0, \quad [\quad,\quad]_2 = \frac{\epsilon^2}{2}\zeta_1 \tag{620}$$

よって,

$$-\frac{dT_1^{(2)}}{d\tau} + T_2^{(2)} = Z_1^{*(2)}, \quad -\frac{dT_2^{(2)}}{d\tau} - T_1^{(2)} + \frac{\epsilon^2}{4}\zeta_1 = Z_2^{*(2)} \tag{621}$$

よって,

$$Z_2^{*(2)} = \frac{\epsilon^2}{8}\zeta_1, \quad T_1^{(2)} = 0, \quad T_2^{(2)} = Z_2^{*(2)} = -\frac{\epsilon^2}{8}\zeta_2 \tag{622}$$

したがって,

$$\frac{d\zeta_1}{dt} = Z_1^{*(0)} + Z_1^{*(1)} + Z_1^{*(2)} = \zeta_2 - \frac{\epsilon}{2}\zeta_1 - \frac{\epsilon^2}{8}\zeta_2 \tag{623}$$

$$\frac{d\zeta_2}{dt} = Z_2^{*(0)} + Z_2^{*(1)} + Z_2^{*(2)} = -\zeta_1 - \frac{\epsilon}{2}\zeta_2 + \frac{\epsilon^2}{8}\zeta_2 \tag{624}$$

解は,

$$\zeta_1 = Ce^{-\frac{\epsilon}{2}t}\cos\left[\left(1 - \frac{\epsilon^2}{8}\right)t + C'\right] \tag{625}$$

$$\zeta_2 = -Ce^{-\frac{\epsilon}{2}t}\sin\left[\left(1 - \frac{\epsilon^2}{8}\right)t + C'\right] \tag{626}$$

$$\begin{aligned} x &= \zeta_1 + D_{T^{(1)}}\zeta_1 + D_{T^{(2)}}\zeta_1 + \frac{1}{2}D_{T^{(2)}}(D_{T^{(1)}}\zeta_1) + \cdots \\ &= \zeta_1 = Ce^{-\frac{\epsilon}{2}t}\cos\left[\left(1 - \frac{\epsilon^2}{8}\right)t + C'\right] \end{aligned} \tag{627}$$

$$\begin{aligned} \dot{x} &= \zeta_2 + D_{T^{(1)}}\zeta_2 + D_{T^{(2)}}\zeta_2 + \frac{1}{2}D_{T^{(1)}}^2\zeta_2 + \cdots \\ &= \zeta_2 - \frac{\epsilon}{2}\zeta_1 - \frac{\epsilon^2}{8}\zeta_2 \\ &= Ce^{-\frac{\epsilon}{2}t}\left\{-\frac{\epsilon}{2}\cos\left[\left(1 - \frac{\epsilon^2}{8}\right)t + C'\right]\right. \\ &\quad \left. - \left(1 - \frac{\epsilon^2}{8}\right)\sin\left[\left(1 - \frac{\epsilon^2}{8}\right)t + C'\right]\right\} \end{aligned} \tag{628}$$

$$\sqrt{1 - \frac{\epsilon^2}{4}} = 1 - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^4) \tag{629}$$

$$\ddot{x} + x + \alpha x^3 = 0$$

これは正準形

$$\frac{dx}{dt} = y = \frac{\partial F}{\partial y} \tag{630}$$

$$\frac{dy}{dt} = -x - \alpha x^3 = -\frac{\partial F}{\partial x} \tag{631}$$

$$F = \frac{1}{2}(y^2 + x^2) + \frac{1}{4}\alpha x^4 \tag{632}$$